POLYNOMIAL SIEVE AND ITS APPLICATION TO Bateman-Horn CONJECTURE AND Goldbach CONJECTURE

ZHANG MINGZHI
Mathematics Department , Sichuan University,
Chengdu , Sichuan Province, 610064 , P.R. of China.
email : mingzhi_zhang1942@yahoo.com

ABSTRACT

For a set $\boldsymbol{E}$ of integers **,** for which $\omega(n) \geq k$, $\forall\, n \in \boldsymbol{E}, \boldsymbol{k} \in \boldsymbol{Z}^{+}$, where $\omega(n)$ is the number of the different prime factors of n, We give a sieve, which can separate the ones for which $\omega(n) = k$ from others .

Applying this sieve to Bateman-Horn conjecture and Goldbach conjecture , we obtain an explicit asymptotic formula with both the main term and the error term . This provides a possible starting point for dealing with some famous problems, Like twin primes, primes of the form $n^2 + a$,Goldbach conjecture and so on .

Key words and phrases : sieve; Bateman-Horn conjecture; Goldbach conjecture ; explicit asymptotic formula; main term; error term; twin primes; primes of the form $n^2 + a$.
2010 Mathematics Subject Classification : 11N32, 11N35, 11P32

1. Introduction

There are a lot of famous problems in number theory which puzzled people in centuries. For example, the infiniteness of the primes of the form $n^2 + 1$,the infiniteness of the twin primes, the infiniteness of the integers n for which $n, n+2, n+6$ are all primes and so on .

G.H.Hardy and J.E.Littlewood [5], [21] had given a series of heuristic asymptotic formulas for the additive problems on primes using circle method.

In 1962 , P.T.Bateman and R.A.Horn [1] had given a famous heuristic asymptotic formula concerning the distribution of primes. A lot of problems are special cases of it. Their conjecture is as follows .

Suppose $P_1, P_2, \dots, P_k \in \boldsymbol{Z}[\mathrm{x}]$ and they satisfy the following three conditions

(a) The leading coefficient of every P_i is positive.

(b) Every P_i is irreducible in $\boldsymbol{Q}[x]$ and no two of them differ by a constant factor .

(c) There is no prime p such that p divides the product $P(n)= P_1(n)P_2(n) \dots P_k(n)$ for all $n \in \boldsymbol{Z}^+$.

Let h_i be the degree of $P_i(n)$ and $Q(P_1, P_2, \dots, P_k; X)$ denote the number of positive integers n $(1 \le n \le X)$ such that $P_1(n)$, $P_2(n), \dots, P_k(n)$ are all primes. Bateman and Horn obtained the following heuristic asymptotic formula by a probabilistic consideration

$$Q(P_1, P_2, \dots, P_k; X) \sim {h_1}^{-1}{h_2}^{-1} \dots {h_k}^{-1} C(P_1, P_2, \dots, P_k) \int_2^X (\log u)^{-k}\, du \quad (1)$$

where

$$C(P_1, P_2, \dots, P_k) = \prod_p \left\{(1 - \frac{\rho(p)}{p})(1 - \frac{1}{p})^{-k}\right\} \quad (2)$$

where $\rho(p)$ denote the number of the solutions of the congruence

$$P(x) \equiv 0 \pmod p \quad (3)$$

and p ranges over all primes .

An upper bound for $Q(P_1, P_2, \dots, P_k; X)$ had been found [16],[17]

$$Q(P_1, P_2, \dots, P_k; X) \le 2^k k!\, C(P_1, P_2, \dots, P_k)\{1 + o(1)\} X/(\log X)^k$$

but, as I known , no nontrivial lower bound has been found .

By similar consideration, we can deal with the following sum

$$S(P_1, P_2, \dots, P_k; X) = \sum_{1 \le n \le X} \Lambda(P_1(n))\Lambda(P_2(n)) \cdots \Lambda(P_k(n)) \quad (4)$$

where $\Lambda(n)$ is the von Mangoldt function and obtain a similar heuristic asymptotic formula

$$S(P_1, P_2, \dots, P_k; X) \sim C(P_1, P_2, \dots, P_k) \cdot X \quad (5)$$

S.Baier [6] proved that (1) and (5) are equivalent .

In this paper , we give a sieve , which can generate an explicit asymptotic formula with both the main term and the error term of

$S(P_1, P_2, \ldots, P_k; X)$. This provides a possible starting point for dealing with some famous problems, Like twin primes, primes of the form $n^2 + 1$ and so on . .

2. An Identity of Arithmetic Function and the Polynomial Sieve

As the analytic form of the Fundamental Theorem of Arithmetic, the Chebyshev's Identity plays an irreplaceable role .

$$\Lambda * 1 = L \tag{6}$$

By differentiation of (6) , we obtain Selberg's identity

$$L\Lambda + \Lambda * \Lambda = L^2 * \mu \tag{7}$$

where μ is the Möbius function.

Starting from (6) and (7) , we can prove the Prime Number Theorem by analytic or elementary methods [13],[14],[15]. Here , we choose a slightly different way.

We denote by e_1 the unit of arithmetic functions and by 1 the function for which $1(n) = 1$ for all $n \in \boldsymbol{Z}^+$. We have

$$\mu * 1 = e_1 \tag{8}$$

By differentiation of (8) and $Le_1 = 0$ we obtain the well-known identity

$$\Lambda = -L\mu * 1 \tag{A_1}$$

or

$$\mu * \Lambda = -L\mu \tag{9}$$

By differentiation of (9) , we obtain

$$\mu * (\Lambda * \Lambda - L\Lambda) = L^2\mu \tag{10}$$

or

$$\Lambda^{*2} = \Lambda * \Lambda = L^2\mu * 1 + L\Lambda \qquad (A_2)$$

By differentiation of (10) further, from (9) and (10) we obtain

$$\mu * (\Lambda^{*3} - 3\Lambda * L\Lambda + L^2\Lambda) = -L^3\mu \qquad (11)$$

or

$$\Lambda^{*3} = \Lambda * \Lambda * \Lambda = -L^3\mu * 1 + 3\Lambda * L\Lambda - L^2\Lambda \qquad (A_3)$$

Generally , by $k-1$ times differentiation of (9) , we obtain

$$\sum_{0\le i\le k-1}\binom{k-1}{i}L^{k-1-i}\mu * L^i\Lambda = -L^k\mu \qquad (12)$$

(12) is a recursive formula for $L^i\mu$, From (9), (10), (11) and (12) we can obtain the following identity by induction

Theorem 1 . For every positive integer k , the following arithmetic function identity holds

$$\Lambda^{*k} = (-1)^k L^k\mu * 1 + B_k \qquad (A_k)$$

where

$$B_k = \sum_{\substack{i_1+i_2+\cdots+i_t+j_1+j_2+\cdots+j_t=k \\ j_1+j_2+\cdots+j_t<k,\, j_r\ge 1\ (1\le r\le t)}} a(k,i_1,i_2,\dots,i_t,j_1,j_2,\dots,j_t) \cdot L^{i_1}\Lambda^{*j_1} * L^{i_2}\Lambda^{*j_2} * \dots * L^{i_t}\Lambda^{*j_t} \qquad (13)$$

where $a(k,i_1,i_2,\dots,i_t,j_1,j_2,\dots,j_t)$ are integers .

Remark . In every term of B_k , the sum of the exponents of Λ is less than k . We denote by $\omega(n)$ the number of the different prime factors of n . If $\omega(n) \ge k$, then $B_k(n) = 0$ by the pigeon hole principle . From (A_k) , we have

$$\Lambda^{*k}(n) = (-1)^k \sum_{d\mid n}\mu(d)(\log d)^k \qquad (14)$$

If $\omega(n) = k$, $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2} \dots {p_k}^{\alpha_k}$, p_i are different primes, from (14) , we obtain

$$k! \cdot \log p_1 \cdot \log p_2 \cdot \ldots \cdot \log p_k$$
$$= \sum_{1 \le i \le k} (-1)^{k+i} \sum_{1 \le j_1 < j_2 < \cdots < j_i \le k} (\log p_{j_1} + \cdots + \log p_{j_i})^k \qquad (15)$$

which is equivalent to the following identity of polynomials in k variables

$$k!\, x_1 x_2 \ldots x_k = \sum_{1 \le i \le k} (-1)^{k+i} \sum_{1 \le j_1 < j_2 < \cdots < j_i \le k} (x_{j_1} + \cdots + x_{j_i})^k \qquad (16)$$

If $\omega(n) = t > k$, then both sides of (14) vanish $([15], p. 47)$. In this case, the equivalent polynomial identity is

$$0 = \sum_{1 \le i \le t} (-1)^i \sum_{1 \le j_1 < j_2 < \cdots < j_i \le t} (x_{j_1} + \cdots + x_{j_i})^k \qquad (17)$$

Therefore, for any set $\boldsymbol{E}$ of integers, for which $\omega(n) \ge k, \forall\, n \in \boldsymbol{E}$, acting (A_k) on the integers of $\boldsymbol{E}$, we can separate the ones for which $\omega(n) = k$ from others. If we consider the sum over $\boldsymbol{E}$, from (14) we get

$$\sum_{n \in E, \omega(n) = k} \Lambda^{*k}(n) = (-1)^k \sum_{n \in E} \sum_{d \mid n} \mu(d) (\log d)^k \qquad (18)$$

So, (18) provides some kind of sieve. We call it polynomial sieve from the discussion above.

Golomb [8] had obtained a result similar to (14).

3. Some lemmas

C. Hooley [12] had given a formula for quadratic polynomials in his study on the sum $\sum_{n \le X} \tau(n^2 + a)$, where $\tau(n)$ is the number of divisors of n. We generalize his result slightly and the proof is almost unchanged.

Lemma 1. (C. Hooley) Let $P(n)$ be a polynomial with integral coefficients, $d \in \mathbf{Z}^+$, $X \in \mathbf{R}^+$ and

$$\mathrm{T}_P(d, X) = \sum_{\substack{P(n) \equiv 0 \ (mod\ d) \\ 1 \le n \le X}} 1 \qquad (19)$$

Then we have

$$T_P(d,X) = X\frac{\rho(d)}{d} + \Psi_P(d,X) - \Phi_P(d) \tag{20}$$

where $\rho(d)$ is the number of the solutions of the congruence $P(x) \equiv 0 \ (mod\ d)$ and

$$\Psi_P(d,X) = \sum_{\substack{P(\nu)\equiv 0\ (mod\ d) \\ 0<\nu\le d}} \psi\left(\frac{X-\nu}{d}\right) \tag{21}$$

$$\Phi_P(d) = \sum_{\substack{P(\nu)\equiv 0\ (mod\ d) \\ 0<\nu\le d}} \psi\left(\frac{-\nu}{d}\right) \tag{22}$$

where $\psi(u) = \frac{1}{2} - \{u\}$, $\{u\}$ is the fraction part of u .

Proof.
$$\begin{aligned} T_P(d,X) &= \sum_{\substack{P(\nu)\equiv 0\ (mod\ d) \\ 0<\nu\le d}} \sum_{\substack{n\equiv \nu\ (mod\ d) \\ 1\le n\le X}} 1 \\ &= \sum_{\substack{P(\nu)\equiv 0\ (mod\ d) \\ 0<\nu\le d}} \left(\left[\frac{X-\nu}{d}\right] + 1\right) \\ &= \sum_{\substack{P(\nu)\equiv 0\ (mod\ d) \\ 0<\nu\le d}} \left(\left[\frac{X-\nu}{d}\right] - \left[\frac{-\nu}{d}\right]\right) \\ &= \sum_{\substack{P(\nu)\equiv 0\ (mod\ d) \\ 0<\nu\le d}} \left(\frac{X}{d} + \psi\left(\frac{X-\nu}{d}\right) - \psi\left(\frac{-\nu}{d}\right)\right) \\ &= X\frac{\rho(d)}{d} + \Psi_P(d,X) - \Phi_P(d) \end{aligned}$$ □

Remark . When $P(n)$ is an even function , the situation becomes simpler and a lot of important problems satisfy this requirement . Let $P(0) = a,$ from $\psi(u) + \psi(-u) = 0, u \notin \boldsymbol{Z}$, we have

$$\Phi_P(d) = \begin{cases} 0 & d \nmid a \\ \frac{1}{2} & d \mid a \end{cases} \tag{23}$$

especially , $\Phi_P(d) = 0$ for $d > a$.

We can generalize the Lemma 1 slightly .

Lemma 2. . Let $P(x)$ be a polynomial with integral coefficients and $m \in \mathbf{Z}^+$, $X \in \mathbf{R}^+$ and

$$\mathrm{T}_P(d,m,X) = \sum_{\substack{P(n)\equiv 0 \ (mod\ d) \\ 1\le n\le X,(P(n),m)=1}} 1$$

Then we have

$$\mathrm{T}_P(d,m,X) =$$
$$\begin{cases} 0 & (d,m)>1 \\ X\frac{\rho(d)}{d}\prod_{p\,|\,m}\left(1-\frac{\rho(p)}{p}\right)+\Psi_P(d,m,X)-\Phi_P(d,m) & (d,m)=1 \end{cases}$$

where

$$\Psi_P(d,m,X) = \sum_{j\,|\,m}\mu(j)\,\Psi_P(dj,X)$$
$$\Phi_P(d,m) = \sum_{j\,|\,m}\mu(j)\ \Phi_P(dj)$$

Proof . It is clear that $(d,m)>1$ imply $\mathrm{T}_P(d,m,X) = 0$. if $(d,m)=1$, then

$$\mathrm{T}_P(d,m,X) = \sum_{\substack{P(n)\equiv 0\ (mod\ d) \\ 1\le n\le X}} \sum_{j\,|\,(P(n),m)}\mu(j)$$
$$= \sum_{j\,|\,m}\mu(j)\sum_{\substack{dj\,|\,(P(n) \\ 1\le n\le X}} 1$$
$$= \sum_{j\,|\,m}\mu(j)(X\frac{\rho(dj)}{dj}+\Psi_P(dj,X)-\Phi_P(dj))$$
$$= X\frac{\rho(d)}{d}\sum_{j\,|\,m}\mu(j)\frac{\rho(j)}{j}+\sum_{j\,|\,m}\mu(j)\,\Psi_P(dj,X)-\sum_{j\,|\,m}\mu(j)\ \Phi_P(dj)$$
$$= X\frac{\rho(d)}{d}\prod_{p\,|\,m}(1-\frac{\rho(p)}{p})+\Psi_P(d,m,X)-\Phi_P(d,m) \qquad \square$$

Suppose $P_1,P_2,\dots,P_k \in \boldsymbol{Z}[x]$ and satisfy the three conditions $(a),(b),(c)$ above. The product in (2) has been proved to be convergent and the coefficient $C(P_1,P_2,\dots,P_k)$ is often called Bateman-Horn constant or Hardy-Littlewood Constant [1],[7],[8],[16]. From the condition (b) , it is easy to know that there is a positive integer m such that if $(P(n)\,,m)=1$, then $P_i(n)$ are pairwise coprime .

K. Conrad [8] obtained a very nice result , he proved unconditionally that the Bateman-Horn constant has another expression.

Lemma 3. (K. Conrad) Suppose $P_1,P_2,\dots,P_k \in \boldsymbol{Z}[x]$ and satisfy the three conditions $(a),(b),(c)$. For any positive integer k and m, the series

$$\sum_{\substack{d\geq 1\\(d,m)=1}} \frac{\mu(d)\rho(d)(\log d)^k}{d} \tag{24}$$

converges and the equality

$$\frac{(-1)^k}{k!}\prod_{p\,\mid\, m}\left(1-\frac{\rho(p)}{p}\right)\sum_{\substack{d\geq 1\\(d,m)=1}} \frac{\mu(d)\rho(d)(\log d)^k}{d} = C(P_1,P_2,\ldots,P_k) \tag{25}$$

holds , where $\rho(d)$ is the number of the solutions of the congruence

$$P(x)= P_1(x)P_2(x)\ldots P_k(x) \equiv 0 \pmod d.$$

4. Generating Bateman-Horn Conjecture

Now , Let us consider the sum (4) . Suppose $P_1, P_2, \ldots, P_k \in \boldsymbol{Z}[x]$ and satisfy the three conditions $(a),(b),(c)$ above and P is the product of $P_i's$ and a positive integer m satisfies that if $(P(n)\,,m)=1$, then $P_i(n)$ are pairwise coprime . From the condition (a) , $\exists\, H \in \boldsymbol{Z}^+$, for $n \geq H$ we have $P_i(n) > 1$ for all i . Hence , $(P(n)\,,m)=1$ and $n \geq H$ imply $\omega\big(P(n)\big) \geq k$. Then , for the set of integers

$$\boldsymbol{E} = \{P(n) \colon H \leq n \leq X,\ (P(n),m)=1\}$$

From (18) , Lemma 2 , Lemma 3 , we have

$$k!\sum_{\substack{H\leq n\leq X\\(P(n),m)=1}} \Lambda(P_1(n))\Lambda(P_2(n))\ldots\Lambda(P_k(n))$$

$$= (-1)^k \sum_{\substack{H\leq n\leq X\\(P(n),m)=1}} \sum_{d\,\mid\, P(n)} \mu(d)(\log d)^k$$

$$= (-1)^k \sum_{\substack{1\leq n\leq X\\(P(n),m)=1}} \sum_{d\,\mid\, P(n)} \mu(d)(\log d)^k + O(1)$$

$$= (-1)^k \sum_{\substack{1<d\leq P(X)\\(d,m)=1}} \mu(d)(\log d)^k \cdot \sum_{\substack{d\,\mid P(n)\\1\leq n\leq X,(P(n),m)=1}} 1 + O(1)$$

$$= (-1)^k \sum_{\substack{1<d\leq P(X)\\(d,m)=1}} \mu(d)(\log d)^k \cdot (X\frac{\rho(d)}{d}\prod_{p\,\mid\, m}\left(1-\frac{\rho(p)}{p}\right)$$

$$+\Psi_P(d,m,X) - \Phi_P(d,m)) + O(1)$$

$$= X\cdot(-1)^k \prod_{p\,\mid\, m}\left(1-\frac{\rho(p)}{p}\right)\sum_{\substack{1<d\leq P(X)\\(d,m)=1}} \frac{\mu(d)\rho(d)(\log d)^k}{d}$$

$$+(-1)^k \sum_{\substack{1<d\leq P(X)\\(d,m)=1}} \mu(d)(\log d)^k \cdot \Psi_P(d,m,X)$$

$$+(-1)^{k+1}\sum_{\substack{1<d\le P(X)\\(d,m)=1}}\mu(d)(\log d)^k\cdot\ \Phi_P(d,m)+O(1)$$

$$=X\cdot k!\,C(P_1,P_2,\dots,P_k)$$

$$+X\cdot(-1)^{k+1}\prod_{p\,\mid\,m}\left(1-\frac{\rho(p)}{p}\right)\sum_{\substack{d>P(X)\\(d,m)=1}}\frac{\mu(d)\rho(d)(\log d)^k}{d}$$

$$+(-1)^k\sum_{\substack{1<d\le P(X)\\(d,m)=1}}\mu(d)(\log d)^k\ \cdot\Psi_P(d,m,X)$$

$$+(-1)^{k+1}\sum_{\substack{1<d\le P(X)\\(d,m)=1}}\mu(d)(\log d)^k\cdot\ \Phi_P(d,m)+O(1)$$

Hence , we obtain Bateman-Horn conjecture with error term.

Theorem 2 . The following explicit asymptotic formula

$$\sum_{\substack{H\le n\le X\\(P(n),m)=1}}\Lambda(P_1(n))\Lambda(P_2(n))\dots\Lambda(P_k(n))$$

$$=X\cdot C(P_1,P_2,\dots,P_k)+\ R_P(m,X) \tag{26}$$

holds , where the error term

$$R_P(m,X)=\ R_{P,1}(m,X)+R_{P,2}(m,X)+R_{P,3}(m,X)+O(1) \tag{27}$$

where

$$R_{P,1}(m,X)=\ \ X\cdot\frac{(-1)^{k+1}}{k!}\prod_{p\,\mid\,m}\left(1-\frac{\rho(p)}{p}\right)\sum_{\substack{d>P(X)\\(d,m)=1}}\frac{\mu(d)\rho(d)(\log d)^k}{d} \tag{28}$$

$$R_{P,2}(m,X)=\ \frac{(-1)^k}{k!}\sum_{\substack{1<d\le P(X)\\(d,m)=1}}\mu(d)(\log d)^k\ \cdot\Psi_P(d,m,X) \tag{29}$$

$$R_{P,3}(m,X)=\ \frac{(-1)^{k+1}}{k!}\sum_{\substack{1<d\le P(X)\\(d,m)=1}}\mu(d)(\log d)^k\ \cdot\Phi_P(d,m) \tag{30}$$

5. Some Remarks

First, we notice that the case $m>1$ can be reduced to the case $m=1$. without losing generality, we can assume m squarefree and Let $m=p_1p_2\dots p_r$. From the condition (c) , $\rho(p_i)$ is less than p_i and there are $t_i=p_i-\rho(p_i)$ residues $(mod\ p_i)$ $0<a_{i,1}<a_{i,2}<\cdots<a_{i,t_i}\le p_i$ which are not the solutions of (3) . Let $u=\prod_{1\le i\le r}t_i$. From the Chinese Remainder Theorem , the systems of the linear congruences

$$\begin{cases} x \equiv a_{1,j_1} (mod\ p_1)\ ,\ \ 1 \le j_1 \le t_1 \\ \qquad \qquad \dots \\ \qquad \qquad \dots \\ x \equiv a_{r,j_r} (\ mod\ p_r)\ ,\ \ 1 \le j_r \le t_r \end{cases} \tag{31}$$

have u solutions $x \equiv b_1, b_2, \dots, b_u\ (\ mod\ m)$ and $n \equiv\ b_j\ (mod\ m)$ imply $(P(n), m) = 1$. Instead of $P(n)$, we can consider the product

$$T_j(n) = P\left(mn + b_j\right) = \ P_1\left(mn + b_j\right)P_2\left(mn + b_j\right) \dots P_k\left(mn + b_j\right), 1 \le j \le\ u$$

which satisfy $\left(T_j(n), m\right) = 1$ for all n and $P_i(mn + b_j)$, $1 \le i \le k$ are pairwise coprime for all n.

Example. Let's consider the set of four primes of the form $n-5,\ n-1,\ n+1,\ n+5$, we have $m = 30$. In order that the values of these four polynomials are pairwise coprime , if and only if n satisfies one of the following systems of linear congruences

$$\begin{cases} n \equiv 0 & (mod\ 2) \\ n \equiv 0 & (mod\ 3) \\ n \equiv\ \pm 2 & (mod\ 5) \end{cases}$$

Their solutions are $n \equiv \pm 12\ (\ mod\ 30)$. Hence, the problem reduced to the following two possible sets

$$\begin{cases} P_1(t) = 30t - 7 \\ P_2(t) = 30t - 11 \\ P_3(t) = 30t - 13 \\ P_4(t) = 30t - 17 \end{cases} \quad \text{and} \quad \begin{cases} P_1(t) = 30t + 17 \\ P_2(t) = 30t + 13 \\ P_3(t) = 30t + 11 \\ P_4(t) = 30t + 7 \end{cases}$$

for which $m = 1$.

Let's consider , for example , the set on the right side. We have

$$\begin{cases} \rho(p) = 0 & p = 2,3,5 \\ \rho(p) = 4 & p > 5 \end{cases}$$

and

$$C(P_1, P_2, P_3, P_4) = \frac{15^4}{4^4} C_4$$

where

$$C_4 = \prod_{p>5} \left(1 - \frac{6p^2 - 4p + 1}{(p-1)^4}\right) \approx 0.62974$$

In this case, the numerical computation shows that the left side of (26) is quite close to the main term .

Secondly, let's consider the error term .

Since m is determined by the polynomial P , so $R_P(m,X)$ is only dependent on P and X in fact .

The term $O(1)$ in (27) is a constant , which is only dependent on the polynomial P and can be easily determined in a single case.

From the convergency of series (24), we have $R_{P,1}(m,X) = o(X)$, when X tends to infinity.

When $P(n)$ is an even function, It's easy to estimate $R_{P,3}(m,X)$. If $d > a = P(0)$, then $\Phi_P(d,m) = 0$, we have

$$R_{P,3}(m,X) = O(\sum_{1<d\le a}(\log d)^k\ ) = O(1)$$

Therefore , in this case , the estimation of $R_P(m,X)$ reduced to the estimation of $R_{P,2}(m,X)$.

6. Examples for $k = 1$

Bateman-Horn conjecture is a quite general conjecture , it has a lot of special cases . First , we consider the case $k = 1$. In this case , $m = 1$ and

$$C(P) = \prod_p \frac{p-\rho(p)}{p-1} \tag{32}$$

For the simplest case $P(n) = n$, we have $\rho(p) = 1,\ C(P) = 1$ and

$$\sum_{1\le n\le X} \Lambda(n) = X + \ R_P(X) \tag{33}$$

where

$$R_P(X) = \sum_{1\le d\le X} \mu(d)\log d \cdot \left\{\frac{X}{d}\right\} + X \cdot \sum_{d>X} \frac{\mu(d)\log d}{d} \tag{34}$$

(33) is another explicit formula for Chebyshev's $\psi(x)$ without resorting the zeros of Riemann zeta function [24]. $R_P(X) = o(X)$ implys Prime Number Theorem and $R_P(X) = O\left(X^{\frac{1}{2}}(\log X)^2\right)$ would imply Riemann Hypothesis [20].

For $P(n) = an + b,\ 0 < b < a\,, (a, b) = 1$, we have $\rho(p) = 1\,, if\ \ p \nmid a$, and $\rho(p) = 0,\ if\ \ p \mid a$, therefore

$$C(P) = \prod_{p \mid a} \frac{p}{p-1} = \frac{a}{\varphi(a)} \tag{35}$$

where $\varphi(n)$ is the Euler's totient function and (26) becomes

$$\sum_{1 \le n \le X} \Lambda(an + b) = \frac{1}{\varphi(a)}(aX + b) + R_P(X) \tag{36}$$

where

$$R_P(X) = X \cdot \sum_{\substack{d > aX+b \\ (d,a)=1}} \frac{\mu(d) logd}{d} - \sum_{\substack{1 < d \le aX+b \\ (d,a)=1}} \mu(d) logd \cdot \psi\left(\frac{X-\nu}{d}\right) + \sum_{\substack{1 < d \le aX+b \\ (d,a)=1}} \mu(d) logd \cdot \psi\left(\frac{-\nu}{d}\right) - \frac{b}{\varphi(a)} \tag{37}$$

where ν is the solution of the congruence $at + b \equiv 0 (mod\ d)$, $0 < \nu \le d$.

(36) provides an explicit formula for a sum over the powers of primes in an arithmetic progressions without resorting the zeros of Dirichlet L- function,[24].

For $P(n) = n^2 + a$, $a \ne -b^2$, $P(n)$ is irreducible in $\mathbf{Q}[\mathrm{x}]$ and is an even function . We have $\rho(2) = 1$ and $\rho(p) = 1 + \left(\frac{-a}{p}\right)$, $p > 2$. Hence

$$\sum_{\substack{1 \le n \le X \\ n^2 + a \ge 1}} \Lambda(n^2 + a)$$

$$= X \cdot \prod_{p>2} \left(1 - \left(\frac{-a}{p}\right) \frac{1}{p-1}\right) + R_{P,1}(X) + R_{P,2}(X) + O(1) \tag{38}$$

where

$$R_{P,2}(X) = -\sum_{1 < d \le X^2 + a} \mu(d) \log d\ \cdot \Psi_P(d, X) \tag{39}$$

As we mentioned above , C. Hooley had investigated the sum $\sum_{n \le X} \tau(n^2 + a)$ related to $P(n) = n^2 + a$ and proved

$$\sum_{1 \le d \le X} \Psi_P(d, X) = O\left(X^{\frac{8}{9}} (\log X)^3\right) \tag{40}$$

from (40) , we can see some hope for proving $R_{P,2}(X) = o(X)$.

For the primes of the form $n^4 + 1$, [3], and generalized Fermat primes of the form $F_{n,t} = n^{2^t} + 1$, [4] , we can get similar results .

7. Examples for $k = 2$

If both $n-1$ and $n+1$ are primes, then we call them twin primes. Generally, we can consider the generalized twin primes, the pair of primes $n-a$ and $n+a$ $(a \geq 1)$. In this case, $P(n) = (n-a)(n+a) = n^2 - a^2$ is an even function and $m = 2a$, $H = a+2$ and $\rho(p) = 1$, $p \mid 2a$; $\rho(p) = 2$, $p \nmid 2a$. Hence

$$C(P_1, P_2) = \prod_{p \mid 2a}(1-\tfrac{1}{p})^{-1} \prod_{p \nmid 2a}(1-\tfrac{2}{p})(1-\tfrac{1}{p})^{-2}$$
$$= 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{\substack{p \mid a \\ p>2}}\frac{p-1}{p-2} \tag{41}$$

and (26) becomes

$$\sum_{\substack{a+2 \leq n \leq X \\ (n+a,2a)=1}} \Lambda(n-a)\Lambda(n+a)$$
$$= X \cdot C(P_1,P_2) + R_{P,1}(2a,X) + R_{P,2}(2a,X) + O(1) \tag{42}$$

where

$$R_{P,1}(2a,X) = -\frac{1}{2}X\prod_{p \mid 2a}\left(1-\frac{1}{p}\right)\sum_{\substack{d > X^2-a^2 \\ (d,2a)=1}}\frac{\mu(d)2^{\omega(d)}(\log d)^2}{d}$$
$$R_{P,2}(2a,X) = \frac{1}{2}\sum_{\substack{1<d\leq X^2-a^2 \\ (d,2a)=1}}\mu(d)(\log d)^2 \cdot \Psi_P(d,2a,X)$$

8. Application to Goldbach Conjecture

Famous Goldbach conjecture on even integers is quite similar to the generalized twin primes problem, but there are some differences also. Let $N = 2X, X \geq 4$ be an even integer, Goldbach guessed N always can be expressed as the sum of two primes, $N = p+q$. If we write $p = X-n$, $1 \leq n \leq X-2$, then $q = X+n$ (We ignore the unique case $p=q$). Therefore, in this case we should consider the polynomial $P(n) = (X-n)(X+n) = X^2-n^2$. $P(n)$ has the negative leading coefficient, it does not satisfy the condition (a) and $P(0) = X^2$ is unbounded. But, since it has positive values for $1 \leq n \leq X-2$, we can also apply (18) and lemma 2 to the integer set

$$\boldsymbol{E} = \{P(n) = X^2 - n^2 : 1 \le n \le X-2, (X+n, 2X) = 1\}$$

In this case, we also have $\rho(p) = 1$, $p \mid 2X$; $\rho(p) = 2$, $p \nmid 2X$. Therefore, we have

$$\sum_{\substack{1 \le n \le X-2 \\ (X+n,2X)=1}} \Lambda(X-n) \cdot \Lambda(X+n)$$

$$= \frac{1}{2!} \sum_{\substack{1 \le n \le X-2 \\ (X+n,2X)=1}} \sum_{d \mid X^2-n^2} \mu(d)(\log d)^2$$

$$= \frac{1}{2} \sum_{\substack{1 < d \le X^2-1 \\ (d,2X)=1}} \mu(d)(\log d)^2 \sum_{\substack{d \mid X^2-n^2 \\ 1 \le n \le X-2 \\ (X+n,2X)=1}} 1$$

$$= \frac{1}{2} \sum_{\substack{1 < d \le X^2-1 \\ (d,2X)=1}} \mu(d)(\log d)^2 \cdot \Big((X-2) \frac{\rho(d)}{d} \prod_{p \mid 2X} \left(1 - \frac{\rho(p)}{p}\right) + \Psi_P(d, 2X, X-2) - \Phi_P(d, 2X)\Big)$$

$$= X \cdot \frac{1}{2} \prod_{p \mid 2X} \left(1 - \frac{1}{p}\right) \sum_{\substack{1 < d \\ (d,2X)=1}} \frac{\mu(d)\rho(d)(\log d)^2}{d} + R_P(2X, X-2) + O(1)$$

$$= X \cdot C(P_1, P_2) + R_{P,1}(2X, X-2) + R_{P,2}(2X, X-2) + R_{P,3}(2X, X-2) + O(1) \tag{43}$$

where

$$R_{P,1}(2X, X-2) = -\frac{X-2}{2} \prod_{p \mid 2X} \left(1 - \frac{1}{p}\right) \sum_{\substack{d > X^2-1 \\ (d,2X)=1}} \frac{\mu(d)\rho(d)(\log d)^2}{d} \tag{44}$$

$$R_{P,2}(2X, X-2) = \frac{1}{2} \sum_{\substack{1 < d \le X^2-1 \\ (d,2X)=1}} \mu(d)(\log d)^2 \cdot \Psi_P(d, 2X, X-2) \tag{45}$$

$$R_{P,3}(2X, X-2) = -\frac{1}{2} \sum_{\substack{1 < d \le X^2-1 \\ (d,2X)=1}} \mu(d)(\log d)^2 \cdot \Phi_P(d, 2X) \tag{46}$$

and $C(P_1, P_2)$ is the same as (41) shows, we need only to replace a in (41) by X .

Since $\Phi_P(d, 2X) = \sum_{j \mid 2X} \mu(j)\, \Phi_P(dj)$ and $P(0) = X^2$, but $dj \nmid X^2$, from (23) we have

$$R_{P,3}(2X, X-2) = 0 \tag{47}$$

So, the problem reduced to the estimation of $R_{P,2}(2X, X-2)$ again.

It has been proved that the asymptotic formula (43) valid for almost all $N = 2X$. $([19], p. 444)$.

8. Conclusion

Up to now , though the numerical computations support Bateman-Horn conjecture strongly in many cases , it is still unproved except the simplest cases $P(n) = n$ and $P(n) = an + b$, $(a, b) = 1$. Goldbach conjecture has been checked up to very large even number, but it is unproved also . Now, from (26) , the problem reduced to the estimation of the error term $R_P(m, X)$. Specially, when $P(n)$ is an even function , the problem reduced to the estimation of $R_{P,2}(m, X)$ further .

As H. Iwaniec noticed that Möbius function $\mu(n)$ has ' Möbius randomness law ' $([18], p. 338)$. In (29) , the factor $\Psi_P(d, m, X)$ is a sum of the values of $\psi(n)$ which has the period 1 and the values in the interval $(-\frac{1}{2}, \frac{1}{2}]$, so we can naturally expect that there is ' $\Psi_P(d, m, X)$ randomness law ' also and there may be a good cancellation in the sum of (29) . Even more, we can expect $R_P(m, X) = o(X)$. If this is the case , then Bateman-Horn conjecture would be proved .

Another possible approach is considering the possible oscillation property of the error term . The result of computation shows the error term may be oscillating and changing signs infinitely often when X tends to infinity . As the difference of a step function and a linear function, the error term is oscillating naturally . If this is the case, then the sum $S(P_1, P_2, \dots, P_k; X)$ would tends to infinity with X , since it is a nondecreasing function of X .

Acknowledgement . I would like to thank the library of Math. Dept.

of Sichuan Univ. and the library of the Chinese Academy of Science for providing references. I would like to thank all authors who published their papers on net, so that I can get a lot of useful information from net.

I would like to thank Prof. A.Granville for his understanding and helps and also to thank Prof. A.Lei for his warm helps.

I would like to thank my family also, a special thanks to my daughter, her consistent support and encouragement help to bring about this paper.

References

[1] P.T.Bateman, R.A.Horn A Heuristic Asymptotic Formula Concerning the Distribution of Prime Numbers, Math. Comp. **16** (1962), 363-367.

[2] D. Shanks On the Conjecture of Hardy & Littlewood Concerning the Number of Primes of the Form n^2+a, Math. Comp. **14** (1960), 321- 332.

[3] D. Shanks On Numbers of the Form n^4+1, Math. Comp. **15** (1961), 186-189.

[4] Y. Gallot A Problem on the Conjecture Concerning the Distribution of Generalized Fermat Prime Numbers, 2003.

[5] G.H. Hardy, J.E. Littlewood Some Problems of 'Partitio Numerorum'; III: On the Expression of a Number as a Sum of Primes, Acta Math., **44 (**1923), 1-70.

[6] S. Baier On the Bateman-Horn Conjecture, J. Number Theory, **96** (2002) 432-448.

[7] T. Foo Chebotarëv Density and the Bateman-Horn Constant, arXiv:1109.3040v4[Math.NT] 4 Oct 2011.

[8] K. Conrad Hardy-Littlewood Constants, in Mathematical Properties of Sequences and Other Combinatorial Structures, Kluwer, 2003, 133-154.

[9] P. Erdös On the Sum $\sum d(f(n))$, J. London Math. Soc. **27** (1952), 7-15.

[10] C. Hooley An Asymptotic Formula in the Theory of Numbers,

Proc. London Math. Soc. Ser.3, **7** (1957), 396-413.
[11] C. Hooley On the Representation of a Number as the Sum of a Square and a Product, Math. Z., **69** (1958), 211-227.
[12] C. Hooley On the Number of Divisors of Quadratic Polynomials, Acta Math. Vol. 110, No.1,(1963), 97-114.
[13] H. G. Diamond Elementary Methods in the Study of the Distribution of Prime Numbers, Bull. Amer. Math. Soc. Vol. 7, No. 3, (1982),553-589.
[14] H.G. Diamond , P. Erdös On Sharp Elementary Prime Number Estimates, L'Enseignement Math. **26** (1980), 313-321.
[15] T.M. Apostol Introduction to Analytic Number Theory , Springer-Verlag, New York , 1976 .
[16] P.T. Bateman , H.G. Diamond Analytic Number Theory, An Introductory Course , World Scientific , New Jersey , 2004 .
[17] G.Greaves Sieves in Number Theory , Springer-Verlag, New York, 2001.
[18] L.G. Hua 数论导引, 科学出版社, 1979.
[19] H. Iwaniec, E. Kowalski Analytic Number Theory, AMS , Providence, Rhode Island, 2004 .
[20] A. Ivić The Riemann Zeta-Function, Theory and Applications, Dover Publications, New York, 1985.
[21] W.Narkiewicz, The Development of Prime Number Theory , Springer-Verlag, Berlin , 2000.
[22] K. Ireland , M. Rosen A Classical Introduction to Modern Number Theory, Springer-Verlag, New York , 1990.
[23] M.B. Nathanson Additive Number Theory, The Classical Bases , Springer-Verlag, New York , 1996 .
[24] H. Davenport, Multiplicative Number Theory, Revised by H. Montgomery, Springer-Verlag, New York, 1980.
[25] C.D. Pan, C.B. Pan 解析数论基础 , 科学出版社, 1999.
[26] Y.Wang Goldbach Conjecture , World Scientific, Singapore , 1984.
[27] Y.Wang 王元论哥德巴赫猜想 , 山东教育出版 社, 1999.

Type equation here.