Sharp bounds on the number of solutions of $X^2 - \left(a^2 + p^{2m}\right) Y^4 = -p^{2m}$

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Background

Problem

Investigate the integer solutions of $aX^2 - bY^4 = c$.

- Why are these interesting equations?
  – squares in binary recurrence sequences
  – quartic model of elliptic curves
- $c = \pm 1, \pm 2, \pm 4$
  Ljunggren, Cohn, Chen-V, Bennett, Walsh, Togbé, Akhtari, Yuan,...
- Other c?
  Walsh observed $X^2 - 2Y^4 = -1$ and $X^2 - 5Y^4 = -4$ are start of a family: $X^2 - \left(2^{2m} + 1\right) Y^4 = -2^{2m}$.
  Together with others showed there are at most three odd solutions.

New Results (I)

Theorem

Let a, m and p be non-negative integers with $a \geq 1$, p a prime, $(a, p^m) = 1$ and $a^2 + p^{2m}$ not a square. Suppose $x^2 - \left(a^2 + p^{2m}\right) y^2 = -1$ has a solution.
There are at most two coprime positive integer solutions of

$$X^2 - \left(a^2 + p^{2m}\right) Y^4 = -p^{2m}.$$

Theorem

Let a, m and p be non-negative integers with $a \geq 1$, p a prime, $(a, 2p^m) = 1$ and $a^2 + 4p^{2m}$ not a square. Suppose $x^2 - \left(a^2 + 4p^{2m}\right) y^2 = -1$ has a solution.
There are at most two coprime positive integer solutions of

$$X^2 - \left(a^2 + 4p^{2m}\right) Y^4 = -4p^{2m}.$$

- Both results are best-possible.

What is special about p^{2m} and $4p^{2m}$?

Lemma

Let a, m and p be non-negative integers with $a \geq 1$, p a prime, $\gcd(a, p^m) = 1$ and $a^2 + p^{2m}$ not a perfect square. Suppose $x^2 - (a^2 + p^{2m})\, y^2 = -1$ has an integer solution.
All coprime integer solutions (x, y) to the quadratic equation $x^2 - (a^2 + p^{2m})\, y^2 = -p^{2m}$ are given by

$$x + y\sqrt{a^2 + p^{2m}} = \pm\left(\pm a + \sqrt{a^2 + p^{2m}}\right)\alpha^{2k},\ k \in \mathbb{Z},$$

where $\alpha = \left(T_1 + U_1\sqrt{a^2 + p^{2m}}\right)/2$ and (T_1, U_1) is the minimum positive solution of the equation $x^2 - (a^2 + p^{2m})\, y^2 = -4$.

- Same holds with p^{2m} replaced by $4p^{2m}$.
- With $b = p^m$ or $2p^m$, any solution with $Y > 1$ has $Y > b^2/2$.

New Results (II)

Definition

Family: all (x, y) *with* $x + y\sqrt{d} \in \left\{\pm\left(\pm e + f\sqrt{d}\right)\alpha^k : k \in \mathbb{Z}\right\}$, *where* α *is a fundamental unit of norm* 1 *in* $\mathbb{Z}\left[\sqrt{d}\right]$ *for* $d, e, f \in \mathbb{Z}$.

Condition 1

Let a *and* b *be positive integers with* $(a, b) = 1$ *and* $a^2 + b^2$ *not a square. Suppose* $x^2 - \left(a^2 + b^2\right) y^2 = -1$ *has a solution and all coprime solutions of* $x^2 - \left(a^2 + b^2\right) y^2 = -b^2$ *are in one family.*

Theorem

If Condition 1 *holds, then there are at most three coprime positive integer solutions of* $X^2 - \left(a^2 + b^2\right) Y^4 = -b^2$.

Representation of solutions

Lemma

If Condition 1 holds and $(X, Y) \neq (a, 1)$ *is a coprime positive integer solution to* $X^2 - \left(a^2 + b^2\right) Y^4 = -b^2$*, then there are* $r, s \in \mathbb{Z}$ *with* $\gcd(r, s) = 1$ *and* $s > r > 0$ *such that*

$$\pm X \pm bi = (a + bi)\,(r \pm si)^4\,. \quad Y = r^2 + s^2.$$

- $x^2 - \left(a^2 + b^2\right) y^2 = -1$ has an integer solution is required here.
- Assume two solutions of $X^2 - \left(a^2 + b^2\right) Y^4 = -b^2$, (X_1, Y_1), (X_2, Y_2) with $Y_2 > Y_1 > b^2/2$.
  Put $x + yi = (r_1 - s_1 i)\,(r_2 - s_2 i)$. Then

$$\left|(X_1 \pm bi)\,(x + yi)^4 - (X_1 \mp bi)\,(x - yi)^4\right| = 2bY_1^4.$$

- Goal: show there are no non-trivial solutions.

Hypergeometric method

- Put

$$X_{n,r}(z) = {}_2F_1(-r - 1/n, -r, 1 - 1/n, z), \quad Y_{n,r} = z^r X_{n,r}\left(z^{-1}\right),$$

  where ${}_2F_1$ denotes the classical hypergeometric function.
- Key relationship:

$$z^{1/n} Y_{n,r}(z) - X_{n,r}(z) = (z-1)^{2r+1} R_{n,r}(z).$$

- $X_{n,r}(z) \in \mathbb{Q}[z]$
- denominators of coefficients of $X_{n,r}(z)$ grow like $c_1(n) c_2(n)^r$.
- $|X_{n,r}(z)| < c_3(n,r) \left|1 + \sqrt{z}\right|^r$ for $|z| \leq 1$.

Effective Irrationality Measures (I)

Lemma

Let $\theta \in \mathbb{C}$ and $\mathbb{K}$ either $\mathbb{Q}$ or an imaginary quadratic field. Suppose that for all non-negative integers r, there are $p_r, q_r \in \mathcal{O}_{\mathbb{K}}$ with $p_r q_{r+1} \neq p_{r+1} q_r$, $|q_r| < k_0 Q^r$ and $|q_r \theta - p_r| \leq \ell_0 E^{-r}$ for real numbers $k_0, \ell_0 > 0$ and $E, Q > 1$. Then for all $p, q \in \mathcal{O}_{\mathbb{K}}$ with $|q| \geq 1/(2\ell_0)$, we have

$$\left| \theta - \frac{p}{q} \right| > \frac{1}{c|q|^{\kappa+1}}, \text{ where } c = 2k_0 Q (2\ell_0 E)^{\kappa} \text{ and } \kappa = \frac{\log Q}{\log E}.$$

- Problem here: wasteful c.
- Cause: $p/q = p_r/q_r$.

Effective Irrationality Measures (II)

Lemma

Let $\theta, E, k_0, \mathbb{K}.\ell_0, p_r, Q$ and q_r be as before. For all $p, q \in \mathcal{O}_{\mathbb{K}}$, let r_0 be the smallest positive integer such that $|q| < E^{r_0}/(2\ell_0)$.

(a) *We have*

$$|q\theta - p| > \frac{1}{2k_0 Q^{r_0+1}}.$$

(b) *When $p/q \neq p_r/q_r$, we have*

$$|q\theta - p| > \frac{1}{2k_0 Q^{r_0}}.$$

- This suggests a method of proof:
  Show no solutions for each value of r_0.
- Here we can take $k_0 = 0.89$, $\ell_0 = 0.4b/X_1$,
  $E = 0.372\sqrt{a^2+b^2}\,Y_1^2/b^2$ and $Q = 10.74\sqrt{a^2+b^2}\,Y_1^2$.

Proof (I)

- For some $\zeta_4 \in \{\pm 1, \pm i\}$, $p = x + iy$. $q = \bar{p}$, Thue equation yields

$$\frac{2b}{\sqrt{a^2+b^2}\,Y_2^2} = \left| \frac{X_1 \pm bi}{X_1 \mp bi} - \left(\frac{p}{q}\right)^4 \right| > 3.7 \left| \left(\frac{X_1 \pm bi}{X_1 \mp bi}\right)^{1/4} - \zeta_4 \frac{p}{q} \right|.$$

- Case 1: $r_0 = 1$ and $\zeta_4 p/q \neq p_1/q_1$:
diophantine lemma implies $Y_2^3 < 372 b^2 Y_1^5$.

Lemma

Suppose Condition 1 holds. Let (X_1, Y_1) and (X_2, Y_2) be two coprime solutions to $X^2 - \left(a^2+b^2\right) Y^4 = -b^2$ with $Y_2 > Y_1 > 1$. Then

$$Y_2 > 7.98 \frac{a^2+b^2}{b^2} Y_1^3.$$

Upper and lower bounds for Y_2 yield contradiction.

- Case 2: $r_0 = 1$ and $\zeta_4 p/q = p_1/q_1$:
similar, but we need to work a bit harder
(use some hypergeometric niceness).

Proof (II)

- Case 3: $r_0 > 1$.

  Our diophantine lemma implies

  $$Y_2^3 < 92b^2 \cdot 1420^{r_0} \left(a^2 + b^2\right)^{r_0} Y_1^{4r_0+5}.$$

  Definition of r_0 implies

  $$Y_2 > 11.25 \cdot 0.138^{r_0} \left(a^2 + b^2\right)^{r_0} b^{2-4r_0} Y_1^{4r_0-1}.$$

  Combining upper and lower bounds for Y_2, we have

  $$a^2 + b^2 < 182,000.$$

  That gives us an upper bound for Y_1 too.
- Complete proof using Pari and MAGMA.

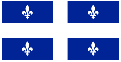
Closing

- Preprint available: https://arxiv.org/abs/1807.04116

Theorem (In progress)

If Condition 1 holds and $a > (29/2)b$, then there are at most two coprime positive integer solutions of $X^2 - \left(a^2 + b^2\right) Y^4 = -b^2$.

- Best possible:
  b a positive integer with $\gcd(b, 10) = 1$ and $a = \left(b^2 - 5\right)/4$.
  $(a, 1)$ and $\left(\left(b^6 + 5b^4 + 15b^2 - 5\right)/16, \left(b^2 + 1\right)/2\right)$ are solutions.



Merci et Thank You!