Numerical evidence for higher-order Stark-type conjectures II: The numerical calculations.

Daniel Vallières

California State University, Chico

Joint work with Kevin McGown (CSU - Chico) and Jonathan Sands (University of Vermont).

CNTA, Quebec City
July 9 - 13, 2018

Numerical evidence for higher order Stark-type conjectures

Brief background: Stark formulated an "imprecise" conjecture for the first non-vanishing Taylor coefficient of an Artin L-function at $s = 0$ (1975) and a precise conjecture for imprimitive abelian L-functions having order of vanishing one at zero (1980). He provided numerical examples for his conjecture as well. Artin L-functions having higher order zeros at $s = 0$ are more complicated to study, but after previous works by Sands, Stark and Tangedal, Stark's conjecture was extended to higher order of vanishing situation by Rubin (1996) and by Popescu (2002).

Numerical evidence for higher order Stark-type conjectures

Brief background: Stark formulated an "imprecise" conjecture for the first non-vanishing Taylor coefficient of an Artin L-function at $s = 0$ (1975) and a precise conjecture for imprimitive abelian L-functions having order of vanishing one at zero (1980). He provided numerical examples for his conjecture as well. Artin L-functions having higher order zeros at $s = 0$ are more complicated to study, but after previous works by Sands, Stark and Tangedal, Stark's conjecture was extended to higher order of vanishing situation by Rubin (1996) and by Popescu (2002).

Numerical evidence for higher order Stark-type conjectures

Brief background: Stark formulated an "imprecise" conjecture for the first non-vanishing Taylor coefficient of an Artin L-function at $s = 0$ (1975) and a precise conjecture for imprimitive abelian L-functions having order of vanishing one at zero (1980). He provided numerical examples for his conjecture as well. Artin L-functions having higher order zeros at $s = 0$ are more complicated to study, but after previous works by Sands, Stark and Tangedal, Stark's conjecture was extended to higher order of vanishing situation by Rubin (1996) and by Popescu (2002).

Numerical evidence for higher order Stark-type conjectures

Brief background: Stark formulated an "imprecise" conjecture for the first non-vanishing Taylor coefficient of an Artin L-function at $s = 0$ (1975) and a precise conjecture for imprimitive abelian L-functions having order of vanishing one at zero (1980). He provided numerical examples for his conjecture as well. Artin L-functions having higher order zeros at $s = 0$ are more complicated to study, but after previous works by Sands, Stark and Tangedal, Stark's conjecture was extended to higher order of vanishing situation by Rubin (1996) and by Popescu (2002).

Brief background: Stark formulated an “imprecise” conjecture for the first non-vanishing Taylor coefficient of an Artin L-function at $s = 0$ (1975) and a precise conjecture for imprimitive abelian L-functions having order of vanishing one at zero (1980). He provided numerical examples for his conjecture as well. Artin L-functions having higher order zeros at $s = 0$ are more complicated to study, but after previous works by Sands, Stark and Tangedal, Stark's conjecture was extended to higher order of vanishing situation by Rubin (1996) and by Popescu (2002).

Motivation: Provide numerical evidence for Rubin's conjecture (1996) and Popescu's conjecture (2002).

Numerical evidence for higher order Stark-type conjectures

Motivation: Provide numerical evidence for Rubin's conjecture (1996) and Popescu's conjecture (2002).

Numerical evidence for higher order Stark-type conjectures

Main tool: Artin systems of S-units. Even though they are not unique, they make the theory concrete.

Main tool: Artin systems of S-units. Even though they are not unique, they make the theory concrete.

Numerical evidence for higher order Stark-type conjectures

Main tool: Artin systems of S-units. Even though they are not unique, they make the theory concrete.

Numerical evidence for higher order Stark-type conjectures

We shall use the same notation as in Sands's talk:

1. K/k is an abelian extension of number fields with Galois group G
2. S is a finite set of places of k containing the infinite ones, the ones that ramify, and r split places, denoted by $v_1, \ldots, v_r$, and such that $|S| \geq r+2$
3. $w_1, \ldots, w_r$ are arbitrarily fixed places of K lying above $v_1, \ldots, v_r$
4. $f : Y_S(K) \longrightarrow E_S(K)$ is an Artin system of S_K-units
5. $\varepsilon_i = f(w_i)$ for $i = 1, \ldots, r$

Numerical evidence for higher order Stark-type conjectures

We shall use the same notation as in Sands's talk:

1. K/k is an abelian extension of number fields with Galois group G
2. S is a finite set of places of k containing the infinite ones, the ones that ramify, and r split places, denoted by $v_1, \ldots, v_r$, and such that $|S| \geq r+2$
3. $w_1, \ldots, w_r$ are arbitrarily fixed places of K lying above $v_1, \ldots, v_r$
4. $f : Y_S(K) \longrightarrow E_S(K)$ is an Artin system of S_K-units
5. $\varepsilon_i = f(w_i)$ for $i = 1, \ldots, r$

Numerical evidence for higher order Stark-type conjectures

We shall use the same notation as in Sands's talk:

1. K/k is an abelian extension of number fields with Galois group G
2. S is a finite set of places of k containing the infinite ones, the ones that ramify, and r split places, denoted by $v_1, \ldots, v_r$, and such that $|S| \geq r+2$
3. $w_1, \ldots, w_r$ are arbitrarily fixed places of K lying above $v_1, \ldots, v_r$
4. $f : Y_S(K) \longrightarrow E_S(K)$ is an Artin system of S_K-units
5. $\varepsilon_i = f(w_i)$ for $i = 1, \ldots, r$

Numerical evidence for higher order Stark-type conjectures

We shall use the same notation as in Sands's talk:

1. K/k is an abelian extension of number fields with Galois group G
2. S is a finite set of places of k containing the infinite ones, the ones that ramify, and r split places, denoted by $v_1, \ldots, v_r$, and such that $|S| \geq r+2$
3. $w_1, \ldots, w_r$ are arbitrarily fixed places of K lying above $v_1, \ldots, v_r$
4. $f : Y_S(K) \longrightarrow E_S(K)$ is an Artin system of S_K-units
5. $\varepsilon_i = f(w_i)$ for $i = 1, \ldots, r$

Numerical evidence for higher order Stark-type conjectures

We shall use the same notation as in Sands's talk:

1. K/k is an abelian extension of number fields with Galois group G
2. S is a finite set of places of k containing the infinite ones, the ones that ramify, and r split places, denoted by $v_1, \ldots, v_r$, and such that $|S| \geq r + 2$
3. $w_1, \ldots, w_r$ are arbitrarily fixed places of K lying above $v_1, \ldots, v_r$
4. $f : Y_S(K) \longrightarrow E_S(K)$ is an Artin system of S_K-units
5. $\varepsilon_i = f(w_i)$ for $i = 1, \ldots, r$

Numerical evidence for higher order Stark-type conjectures

We shall use the same notation as in Sands's talk:

1. K/k is an abelian extension of number fields with Galois group G
2. S is a finite set of places of k containing the infinite ones, the ones that ramify, and r split places, denoted by $v_1, \ldots, v_r$, and such that $|S| \geq r+2$
3. $w_1, \ldots, w_r$ are arbitrarily fixed places of K lying above $v_1, \ldots, v_r$
4. $f : Y_S(K) \longrightarrow E_S(K)$ is an Artin system of S_K-units
5. $\varepsilon_i = f(w_i)$ for $i = 1, \ldots, r$

Numerical evidence for higher order Stark-type conjectures

After an Artin system of S_K-units has been found, the evaluator η which is the unique preimage of $\theta_S^{(r)}(0) \in \mathbb{C}[G]$ under the isomorphism (on appropriate components)

$$u_1 \wedge u_2 \wedge \cdots \wedge u_r \mapsto \det\left(-\sum_{\sigma \in G} \log|u_i^{\sigma}|_j \cdot \sigma^{-1}\right)$$

is given by

$$\eta = \beta_S(f) \cdot \varepsilon_1 \wedge \ldots \wedge \varepsilon_r \in \mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} E_S(K),$$

where

$$\beta_S(f) = \theta_S^{(r)}(0)/\mathrm{Reg}(U_f) \in \mathbb{C}[G]$$

as explained in Sands's talk. (Recall that $U_f = f(X_S(K))$.)

Numerical evidence for higher order Stark-type conjectures

After an Artin system of S_K-units has been found, the evaluator η which is the unique preimage of $\theta_S^{(r)}(0) \in \mathbb{C}[G]$ under the isomorphism (on appropriate components)

$$u_1 \wedge u_2 \wedge \cdots \wedge u_r \mapsto \det\left(-\sum_{\sigma \in G} \log |u_i^{\sigma}|_j \cdot \sigma^{-1}\right)$$

is given by

$$\eta = \beta_S(f) \cdot \varepsilon_1 \wedge \ldots \wedge \varepsilon_r \in \mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} E_S(K),$$

where

$$\beta_S(f) = \theta_S^{(r)}(0)/\mathrm{Reg}(U_f) \in \mathbb{C}[G]$$

as explained in Sands's talk. (Recall that $U_f = f(X_S(K))$.)

After an Artin system of S_K-units has been found, the evaluator η which is the unique preimage of $\theta_S^{(r)}(0) \in \mathbb{C}[G]$ under the isomorphism (on appropriate components)

$$u_1 \wedge u_2 \wedge \cdots \wedge u_r \mapsto \det\left(-\sum_{\sigma \in G} \log |u_i^{\sigma}|_j \cdot \sigma^{-1}\right)$$

is given by

$$\eta = \beta_S(f) \cdot \varepsilon_1 \wedge \ldots \wedge \varepsilon_r \in \mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} E_S(K),$$

where

$$\beta_S(f) = \theta_S^{(r)}(0)/\mathrm{Reg}(U_f) \in \mathbb{C}[G]$$

as explained in Sands's talk. (Recall that $U_f = f(X_S(K))$.)

Numerical evidence for higher order Stark-type conjectures

The special value $\theta_S^{(r)}(0)$ can be calculated with PARI, and the χ-components of the regulator $\mathrm{Reg}(U_f)$ can be made concrete:

Theorem

If $|S| \geq r+1$ and χ is non-trivial and such that the corresponding imprimitive L-function has precisely order of vanishing r at zero, then the χ-component of $\mathrm{Reg}(U_f)$ is given by

$$\det\left(-\sum_{\sigma\in G}\chi(\sigma)\log|\varepsilon_i^{\sigma}|_j\right).$$

Numerical evidence for higher order Stark-type conjectures

The special value $\theta_S^{(r)}(0)$ can be calculated with PARI, and the χ-components of the regulator $\mathrm{Reg}(U_f)$ can be made concrete:

Theorem

If $|S| \geq r+1$ and χ is non-trivial and such that the corresponding imprimitive L-function has precisely order of vanishing r at zero, then the χ-component of $\mathrm{Reg}(U_f)$ is given by

$$\det\left(-\sum_{\sigma\in G}\chi(\sigma)\log|\varepsilon_i^{\sigma}|_j\right).$$

Numerical evidence for higher order Stark-type conjectures

Thus, the element

$$\beta_S(f) \in \mathbb{C}[G]$$

can be calculated numerically. Stark's conjecture over $\mathbb{Q}$ predicts that

$$\beta_S(f) \in \mathbb{Q}[G],$$

and with enough precision, those rational numbers can be found numerically. But what are those rational numbers?

Numerical evidence for higher order Stark-type conjectures

Thus, the element

$$\beta_S(f) \in \mathbb{C}[G]$$

can be calculated numerically. Stark's conjecture over $\mathbb{Q}$ predicts that

$$\beta_S(f) \in \mathbb{Q}[G],$$

and with enough precision, those rational numbers can be found numerically. But what are those rational numbers?

Numerical evidence for higher order Stark-type conjectures

Thus, the element

$$\beta_S(f) \in \mathbb{C}[G]$$

can be calculated numerically. Stark's conjecture over $\mathbb{Q}$ predicts that

$$\beta_S(f) \in \mathbb{Q}[G],$$

and with enough precision, those rational numbers can be found numerically. But what are those rational numbers?

Numerical evidence for higher order Stark-type conjectures

Burns formulated the following conjecture (2011):

Conjecture (Burns)

With the same hypotheses as above, one has

1. $w_K \cdot m^r \cdot \beta_S(f) \in \mathbb{Z}[G]$
2. $w_K \cdot m^r \cdot \beta_S(f) \in \mathrm{Ann}_{\mathbb{Z}[G]} Cl_S(K)$,

where w_K is the number of roots of unity in K, and m is the index

$$(E_S(K) : \mu(K) \cdot f(X_S(K)).$$

Numerical evidence for higher order Stark-type conjectures

Burns formulated the following conjecture (2011):

Conjecture (Burns)

With the same hypotheses as above, one has

1. $w_K \cdot m^r \cdot \beta_S(f) \in \mathbb{Z}[G]$
2. $w_K \cdot m^r \cdot \beta_S(f) \in \mathrm{Ann}_{\mathbb{Z}[G]} Cl_S(K)$,

where w_K is the number of roots of unity in K, and m is the index

$$(E_S(K) : \mu(K) \cdot f(X_S(K))).$$

Numerical evidence for higher order Stark-type conjectures

Burns formulated the following conjecture (2011):

Conjecture (Burns)

With the same hypotheses as above, one has

1. $w_K \cdot m^r \cdot \beta_S(f) \in \mathbb{Z}[G]$
2. $w_K \cdot m^r \cdot \beta_S(f) \in \mathrm{Ann}_{\mathbb{Z}[G]} Cl_S(K)$,

where w_K is the number of roots of unity in K, and m is the index

$$(E_S(K) : \mu(K) \cdot f(X_S(K))).$$

Numerical evidence for higher order Stark-type conjectures

Burns formulated the following conjecture (2011):

Conjecture (Burns)

With the same hypotheses as above, one has

1. $w_K \cdot m^r \cdot \beta_S(f) \in \mathbb{Z}[G]$
2. $w_K \cdot m^r \cdot \beta_S(f) \in \mathrm{Ann}_{\mathbb{Z}[G]} Cl_S(K)$,

where w_K is the number of roots of unity in K, and m is the index

$$(E_S(K) : \mu(K) \cdot f(X_S(K)).$$

Numerical evidence for higher order Stark-type conjectures

Burns formulated the following conjecture (2011):

Conjecture (Burns)

With the same hypotheses as above, one has

1. $w_K \cdot m^r \cdot \beta_S(f) \in \mathbb{Z}[G]$
2. $w_K \cdot m^r \cdot \beta_S(f) \in \mathrm{Ann}_{\mathbb{Z}[G]} Cl_S(K)$,

where w_K is the number of roots of unity in K, and m is the index

$$(E_S(K) : \mu(K) \cdot f(X_S(K))).$$

Numerical evidence for higher order Stark-type conjectures

Part one of Burns's conjecture gives a conjectural bound on the denominators for the rational numbers showing up in

$$\beta_S(f).$$

So, if the integer d integralizes $\beta_S(f)$, then Popescu's conjecture predicts that for $\phi_1, \ldots, \phi_{r-1} \in \mathrm{Hom}_{\mathbb{Z}[G]}(E_S(K), \mathbb{Z}[G])$, the S_K-unit

$$w_K \cdot d \cdot \beta_S(f) \cdot \phi_1 \wedge \ldots \wedge \phi_{r-1}(\varepsilon_1 \wedge \ldots \wedge \varepsilon_r)$$

is the dth power of an S_K-unit ε satisfying $K(\varepsilon^{1/w_K})/k$ is abelian.

Part one of Burns's conjecture gives a conjectural bound on the denominators for the rational numbers showing up in

$$\beta_S(f).$$

So, if the integer d integralizes $\beta_S(f)$, then Popescu's conjecture predicts that for $\phi_1, \ldots, \phi_{r-1} \in \mathrm{Hom}_{\mathbb{Z}[G]}(E_S(K), \mathbb{Z}[G])$, the S_K-unit

$$w_K \cdot d \cdot \beta_S(f) \cdot \phi_1 \wedge \ldots \wedge \phi_{r-1}(\varepsilon_1 \wedge \ldots \wedge \varepsilon_r)$$

is the dth power of an S_K-unit ε satisfying $K(\varepsilon^{1/w_K})/k$ is abelian.

Numerical evidence for higher order Stark-type conjectures

Then using the isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(E_S(K), \mathbb{Z}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{Z}[G]}(E_S(K), \mathbb{Z}[G]),$$

given by

$$f \mapsto \left(u \mapsto \sum_{\sigma \in G} f(\sigma^{-1} u) \cdot \sigma \right)$$

one can use PARI to check Popescu's conjecture. (It involves finding a $\mathbb{Z}$-basis for $E_S(K)$ and doing some linear algebra over $\mathbb{Z}$ ...)

Numerical evidence for higher order Stark-type conjectures

Popescu's conjecture is known via the ETNC when the base field is $\mathbb{Q}$ (Burns-Greither, 2003, Flach, 2011, and Burns, 2007) and partially known when the base field is quadratic imaginary (Bley, 2006).

Numerical evidence for higher order Stark-type conjectures

We decided to check the conjectures above in the setting where K/k is a cubic abelian extension of totally real fields with k quadratic over $\mathbb{Q}$. Here $r = 2$ and the split places are the two real places of the base field, so we are in an order of vanishing two situation. In this setup, Popescu's conjecture is actually equivalent to Rubin's conjecture, since $\mu(K)$ is cohomologically trivial.

Numerical evidence for higher order Stark-type conjectures

We decided to check the conjectures above in the setting where K/k is a cubic abelian extension of totally real fields with k quadratic over $\mathbb{Q}$. Here $r = 2$ and the split places are the two real places of the base field, so we are in an order of vanishing two situation. In this setup, Popescu's conjecture is actually equivalent to Rubin's conjecture, since $\mu(K)$ is cohomologically trivial.

Numerical evidence for higher order Stark-type conjectures

We decided to check the conjectures above in the setting where K/k is a cubic abelian extension of totally real fields with k quadratic over $\mathbb{Q}$. Here $r = 2$ and the split places are the two real places of the base field, so we are in an order of vanishing two situation. In this setup, Popescu's conjecture is actually equivalent to Rubin's conjecture, since $\mu(K)$ is cohomologically trivial.

Numerical evidence for higher order Stark-type conjectures

We ran our algorithm briefly explained above over all such extensions where K/k is ramified and $\Delta_K \leq 10^{12}$ for a total of 19197 examples. Every single time, Popescu's conjecture was satisfied.

Numerical evidence for higher order Stark-type conjectures

We ran our algorithm briefly explained above over all such extensions where K/k is ramified and $\Delta_K \leq 10^{12}$ for a total of 19197 examples. Every single time, Popescu's conjecture was satisfied.

Numerical evidence for higher order Stark-type conjectures

In all these examples, we also checked numerically Burns's conjecture. But here something even better happened for which we do not know any reason. We always had

$$w_K \cdot m \cdot \beta_S(f) \in \mathbb{Z}[G]$$

rather than

$$w_K \cdot m^2 \cdot \beta_S(f) \in \mathbb{Z}[G]$$

as expected by Burns's conjecture. Furthermore,

$$w_K \cdot m \cdot \beta_S(f) \in \mathrm{Ann}_{\mathbb{Z}[G]}(Cl_S(K)).$$

Numerical evidence for higher order Stark-type conjectures

In all these examples, we also checked numerically Burns's conjecture. But here something even better happened for which we do not know any reason. We always had

$$w_K \cdot m \cdot \beta_S(f) \in \mathbb{Z}[G]$$

rather than

$$w_K \cdot m^2 \cdot \beta_S(f) \in \mathbb{Z}[G]$$

as expected by Burns's conjecture. Furthermore,

$$w_K \cdot m \cdot \beta_S(f) \in \mathrm{Ann}_{\mathbb{Z}[G]}(Cl_S(K)).$$

Numerical evidence for higher order Stark-type conjectures

In all these examples, we also checked numerically Burns's conjecture. But here something even better happened for which we do not know any reason. We always had

$$w_K \cdot m \cdot \beta_S(f) \in \mathbb{Z}[G]$$

rather than

$$w_K \cdot m^2 \cdot \beta_S(f) \in \mathbb{Z}[G]$$

as expected by Burns's conjecture. Furthermore,

$$w_K \cdot m \cdot \beta_S(f) \in \mathrm{Ann}_{\mathbb{Z}[G]}(Cl_S(K)).$$

Numerical evidence for higher order Stark-type conjectures

In all these examples, we also checked numerically Burns's conjecture. But here something even better happened for which we do not know any reason. We always had

$$w_K \cdot m \cdot \beta_S(f) \in \mathbb{Z}[G]$$

rather than

$$w_K \cdot m^2 \cdot \beta_S(f) \in \mathbb{Z}[G]$$

as expected by Burns's conjecture. Furthermore,

$$w_K \cdot m \cdot \beta_S(f) \in \mathrm{Ann}_{\mathbb{Z}[G]}(Cl_S(K)).$$

Numerical evidence for higher order Stark-type conjectures

In all these examples, we also checked numerically Burns's conjecture. But here something even better happened for which we do not know any reason. We always had

$$w_K \cdot m \cdot \beta_S(f) \in \mathbb{Z}[G]$$

rather than

$$w_K \cdot m^2 \cdot \beta_S(f) \in \mathbb{Z}[G]$$

as expected by Burns's conjecture. Furthermore,

$$w_K \cdot m \cdot \beta_S(f) \in \mathrm{Ann}_{\mathbb{Z}[G]}(Cl_S(K)).$$

Thank you!