Numerical evidence for higher order Stark-type conjectures I: Theory.

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Joint work with Kevin McGown and Daniel Vallières
California State University, Chico.
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Numerical evidence for higher order Stark-type conjectures

The aim of the conjectures is to understand the precise relationship between the arithmetic invariants of a number field K and the analytic L-functions of K.

First: The Arithmetic Side: Number Fields

Let: K/k be an abelian extension of number fields

G be the abelian Galois group

S_∞ be the set of infinite places of k

S be a finite set of places of k containing S_∞ and the primes that ramify in K.

$v_1, v_2, \ldots v_r$ be primes in S that split completely in K. We will assume the cardinalities satisfy $|S| > r + 1$.

S_K be the set of places of K above those in S.

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Also let
$Y_S(K)$ denote the free abelian group on S_K. It is a $\mathbb{Z}[G]$-module.
$X_S(K)$ denote the $\mathbb{Z}[G]$-submodule of elements whose coefficients sum to 0.
$E_S(K)$ denote the group of S_K-units of K, also a $\mathbb{Z}[G]$-module.

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The Arithmetic Side: Artin S-units

Definition

An Artin system of S_K-units $\mathcal{A}$ is a collection of S_K-units

$$\mathcal{A} = \{\varepsilon_w \,|\, w \in S_K\} \subseteq E_S(K),$$

such that the group morphism

$$f : Y_S(K) \longrightarrow E_S(K)$$

defined by $\mathbb{Z}$-linearly extending $w \mapsto \varepsilon_w$ satisfies the following properties:

1. *f is G-equivariant,*
2. $\ker(f) = \mathbb{Z} \cdot \alpha$ *for some* $\alpha \in Y_S(K)$, $\alpha \notin X_S(K)$.

Consequently $U_f = f(X_S(K))$ is of finite index in $E_S(K)$.
We set $m = (E_S(K) : \mu(K) \cdot U_f)$, where $\mu(K)$ denotes the roots of unity in K.

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Theorem

Artin S-units exist.

Proof.

Modify Artin's proof for S_∞-units, based on the familiar proof of the Dirichlet unit theorem. □

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The Arithmetic Side: Logarithmic Map

For $u \in E_S(K)$, let $|u|_w$ denote the normalized valuation of u at w and let

$$\lambda(u) = -\sum_{w \in S_K} \log |u|_w \cdot w \in \mathbb{C}Y_S(K)$$

Tensoring up with $\mathbb{C}$, let $f_{\mathbb{C}}$ be the $\mathbb{C}$-linear extension of f so that

$$f_{\mathbb{C}} \circ \lambda : \mathbb{C}E_S(K) \to \mathbb{C}E_S(K)$$

and define the regulator of $U_f = f(X_S(K))$ in $\mathbb{C}[G]$ as

$$\mathrm{Reg}(U_f) = \det_{\mathbb{C}[G]}(f_{\mathbb{C}} \circ \lambda) \in \mathbb{C}[G]$$

Each irreducible character χ on G has an associated idempotent e_χ, and $\mathbb{C}[G] \cong \oplus_\chi \mathbb{C}e_\chi$.
The map above is $\mathbb{C}[G]$-linear so that the determinant can be evaluated component-wise.

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The Analytic Side: L-functions

Letting $\sigma_\mathfrak{p}$ denote the Frobenius automorphism in G for $\mathfrak{p} \notin S$, the abelian L-function for a (irreducible) character χ of G is

$$L_S(s, \chi) = \prod_{\mathfrak{p} \notin S} (1 - N(\mathfrak{p})^{-s} \chi(\sigma_\mathfrak{p}))^{-1}$$

which extends to a meromorphic function on $\mathbb{C}$.

If r is the number of places of S that split completely in K, then $L_S(s, \chi)$ vanishes to order at least r at $s = 0$.

One considers the rth Maclaurin coefficient $L_S^{(r)}(0, \chi)$ and defines the Stickelberger element

$$\theta_S^{(r)}(0) = \sum_{\chi} L_S^{(r)}(0, \chi) e_{\overline{\chi}} \in \mathbb{C}[G]$$

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Analytic and Arithmetic: Stark's Rationality Conjecture

Put

$$\beta_S(f) = \theta_S^{(r)}(0)/\operatorname{Reg}(U_f) \in \mathbb{C}[G]$$

where the quotient can be taken componentwise in $\mathbb{C}[G] \cong \oplus_\chi \mathbb{C}e_\chi$.
Notice that for $K = k$ with appropriate hypotheses on S, this would relate to the S-regulator $R_S(K)$ and S-class number $h_S(K)$ in the form

$$\frac{\zeta_{S,K}^{(r)}(0)}{\operatorname{Reg}(U_f)} = \frac{-h_S(K)R_S(K)}{w_K \operatorname{Reg}(U_f)} = \frac{-h_S(K)}{w_K(E_S(K) : \mu(K)U_f)} \in \mathbb{Q}$$

A form of Stark's rationality conjecture in our situation is

Conjecture

$\beta_S(f) \in \mathbb{Q}[G]$.

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Analytic and Arithmetic: The L-function evaluator η

For each j from 1 to r, fix a choice of w_j in S_K over the split prime v_j and let $| \;|_j$ denote the associated normalized absolute value on K.

Also set $\epsilon_j = f(w_j)$, the corresponding Artin unit. One defines

$$\eta = \beta_S(f) \cdot \epsilon_1 \wedge \epsilon_2 \wedge \cdots \wedge \epsilon_r \in \mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} E_S(K).$$

Theorem

Up to sign, the element η is independent of the choice of Artin units and is characterized as the unique preimage of $\theta_S^{(r)}(0) \in \mathbb{C}[G]$ under the isomorphism (on appropriate components) that sends $u_1 \wedge u_2 \wedge \cdots \wedge u_r$ to $\det(-\sum_{\sigma \in G} \log |u_i^\sigma|_j \cdot \sigma^{-1})$.
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Analytic and Arithmetic: Popescu's Integrality Conjecture

Conjecture

(Popescu) For all choices of $r-1$ $\mathbb{Z}[G]$-module homomorphisms $\phi_i : E_S(K) \to \mathbb{Z}[G]$, the image of $w_K\eta$ under $\phi_1 \wedge \cdots \wedge \phi_{r-1}$ in $\mathbb{C}E_S(K)$ is represented by an element ϵ such that

1. $\epsilon \in E_S(K)$
2. $K(\epsilon^{1/w_K})/k$ *is abelian Galois.*

When $r = 1$, this is equivalent to Stark's refined abelian rank one conjecture, and ϵ is then known as the "Stark unit".
A weaker form of the conjecture may be illuminating. It states that, with one more ϕ, the image of $w_K\eta$ under $\phi_1 \wedge \cdots \wedge \phi_r$ in $\mathbb{C}[G]$ is

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Analytic and Arithmetic: Burns' Annihilation Conjecture

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(Burns) With $m = (E_S(K) : \mu(K) \cdot U_f)$ *as above,*

$$w_K m^r \beta_S(f) \in \mathbb{Z}[G]$$

and furthermore, this element annihilates the S*-class group* $\mathcal{C}\ell_S(K)$ *of* K*.*

This conjecture and Popescu's conjecture are both consequences of the more general Leading Term Conjecture of Burns, which Burns, Flach and Greither have proved for K abelian over $\mathbb{Q}$, and Johnston and Nickel have proved when $K/\mathbb{Q}$ is an S_3-extension, assuming Leopoldt's conjecture.

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