Automorphism groups of K3 surfaces over nonclosed fields

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CNTA XV, July 12, 2018

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A standard conjecture

A *K3 surface* is a surface S with K_S and $\pi_1(S)$ both trivial. Examples: a smooth quartic in $\mathbb{P}^3$, the smooth intersection of three quadrics in $\mathbb{P}^5$, etc. The following statement is generally believed:

Conjecture

Let S be a K3 surface over a number field K. Then $S(K)$ is either empty or Zariski dense.

$S(K)$ can often be proved to be empty by showing that $S(K_\mathfrak{p})$ is empty, where $K_\mathfrak{p}$ is the completion of K at some place. The *Brauer obstruction* can also be used (and it is conjectured that if $S(K)$ is empty, one of these is responsible).

Proving density

The easiest way to prove that $S(K)$ is Zariski dense is to show that S has infinitely many rational curves over K.

Suppose that S has an elliptic fibration $\pi : S \to \mathbb{P}^1$ of positive rank. Then there are infinitely many sections.

If S contains a single rational curve C, we can try to show that the $\mathrm{Aut}\, S$-orbit of C is infinite.

Since $\mathrm{Aut}\, S$ acts on $\mathrm{Pic}\, S$ with finite kernel, this can be reduced to linear algebra.

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Our goal

Let S be a K3 surface over a field k, such as $\mathbb{Q}$.

We would like to describe $\operatorname{Aut} S$ in terms of $\operatorname{Aut} S/\bar{k}$ and the embedding of $\operatorname{Pic} S/k$ into $\operatorname{Pic} S/\bar{k}$.

(At least theoretically: the description of $\operatorname{Aut} S/\bar{k}$ is not always easy to work with.)

The automorphism group over $\bar{k}$

Let S be a K3 surface over an algebraically closed field and let Λ be its Picard lattice. Then $\mathrm{Aut}\, S$ acts on Λ with finite kernel.

Let $c \in \Lambda$ be such that $(c, c) = -2$. Then there is a reflection $\rho_c : x \to x + (x, c)c \in O(\Lambda)$. Let the group generated by these be $W(\Lambda)$.

The map $\mathrm{Aut}\, S \to O(\Lambda)/W(\Lambda)$ has finite kernel and cokernel (Shafarevich, Piatetski-Shapiro).

Invariants

Let S be a K3 surface over a field k, and let $\bar{S} = S \otimes_k \bar{k}$. Then $\mathrm{Gal}(k)$ acts on $\mathrm{Pic}\,\bar{S}$ and so on $W(\mathrm{Pic}\,\bar{S})$ by permuting the generators. It also acts on $\mathrm{Aut}\,\bar{S}$ by its action on the coefficients of polynomials defining automorphisms and on $\mathrm{Aut}(\mathrm{Pic}\,\bar{S})$ by conjugation.

Now, $\mathrm{Aut}(\mathrm{Pic}\,\bar{S})^{\mathrm{Gal}(k)}$ acts on $(\mathrm{Pic}\,\bar{S})^{\mathrm{Gal}(k)}$. Therefore a subgroup R_S of finite index in $W(\mathrm{Pic}\,\bar{S})^{\mathrm{Gal}(k)}$ acts on $\mathrm{Pic}\,S$ (because $\mathrm{Pic}\,S$ has finite index in $(\mathrm{Pic}\,\bar{S})^{\mathrm{Gal}(k)}$).

The main result

We thus have a map $R_S \to \mathrm{Aut}\,\mathrm{Pic}\,S$. The following is our main theorem:

Theorem

The natural map $\mathrm{Aut}\,S \to \mathrm{Aut}\,\mathrm{Pic}\,S/R_S$ *has finite kernel and cokernel.*

(Note that if k is algebraically closed, then $R_S = W(\mathrm{Pic}\,S)$ and we recover the previous result.)

A few words about the proof

The main ingredients are the theory over algebraically closed fields, some straightforward arguments on lattices, and the following lemma:

Lemma

Let G be a group acting on groups A, B (not necessarily commutative) such that $A \to B$ has finite kernel and cokernel. Then $A^G \to B^G$ has finite kernel and cokernel.

For arithmetical applications one needs to know more about R_S than just its definition.

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Coxeter groups

A *Coxeter group* is a pair (G, R), where G is a group generated by a set R of elements of order 2 with all relations of the form $(r_i r_j)^{n_{ij}} = 1$. In particular, groups generated by reflections (such as $W_{\bar{S}}$) are Coxeter groups. We set $n_{ij} = \infty$ if there is no relation involving r_i, r_j.

An element in a Coxeter group has a *length*. If G is finite, the element of maximal length is unique.

A Coxeter group is represented by a *Coxeter diagram*, which is a graph whose vertices are the elements of R and an edge labeled n_{ij} joins the vertices R_i, R_j if $n_{ij} < \infty$.

Actions on Coxeter groups

If Γ is a group of permutations of R such that $n_{ij} = n_{i^\gamma j^\gamma}$ for all i, j, γ, then Γ extends to an action on G.

We use the following theorem, due to Hée but more accessibly proved by Geck-Iancu:

Theorem

Let (G, R) be a Coxeter group and Γ a group of permutations of R as before. For each orbit O of Γ, let (G_O, O) be the Coxeter group generated by r_i for $i \in O$, and let $\mathcal{L}$ be the set of longest words in G_O for O with G_O finite. Then $(G^\Gamma, \mathcal{L})$ is a Coxeter group.

Understanding R_S

Recall that R_S was defined to be the maximal subgroup of $W(\mathrm{Pic}\, S)^{\mathrm{Gal}(k)}$ that acts on $\mathrm{Pic}\, S$. (It is of finite index.)

Some generators of $W(\mathrm{Pic}\, S)^{\mathrm{Gal}(k)}$ are easy to describe. For example, let $C_1, \ldots, C_n$ be disjoint and Galois-conjugate -2-curves. Then the ρ_{C_i} (reflections in the C_i) commute, so the only element of $\langle \rho_{C_i} \rangle$ fixed by Galois is $\prod_{i=1}^{n} \rho_{C_i}$.

The only other case

If C, D are conjugate and intersect in 1 point, then $\rho_C \rho_D \rho_C$ is fixed by Galois.

More generally, let the orbit consist of $2n$ curves $C_1, D_1, \ldots, C_n, D_n$, where $C_i \cdot D_i = 1$ and all other intersections are 0. Then $\prod_{i=1}^n \rho_{C_i} \rho_{D_i} \rho_{C_i}$ is fixed by Galois; this corresponds to $n_{\rho_{C_1} \rho_{D_1}} = 3$.

There are no other examples. If the Coxeter diagram of a finite Coxeter group has a transitive group of automorphisms, every connected component has order at most 2. But we must have $n_{ij} \in \{2, 3, \infty\}$ for the intersection numbers to be integers.

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Basic remarks

Over $\mathbb{C}$ the automorphism group of S depends (up to finite index) only on $\mathrm{Pic}\, S$.

Our primary interest (at least for now) is to distinguish K3 surfaces with $\mathrm{Aut}\, S$ finite from those with $\mathrm{Aut}\, S$ infinite.

We will give two examples to illustrate the difference between the situation with k algebraically closed and with k not.

Example 1

Let $S/\mathbb{Q}$ be a K3 surface that, over $\bar{\mathbb{Q}}$, admits an elliptic fibration of rank 0 with four conjugate fibres of type I_2 or II and no other reducible fibres. Then $\operatorname{Pic} S$ has Gram matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -8 \end{pmatrix}$.

A K3 surface over $\bar{\mathbb{Q}}$ with this Picard lattice has an elliptic fibration of rank 1 and hence an infinite automorphism group.

However, $\operatorname{Pic} S/\bar{\mathbb{Q}}$ has the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus -2I_4$.

Nikulin showed that a K3 surface over an algebraically closed field with this Picard lattice has finite automorphism group.

Example 2

In fact one can give an example of a K3 surface $S/\mathbb{Q}$ where $\operatorname{Aut} S$ is finite even though for all extensions $K/\mathbb{Q}$, a K3 surface over $\mathbb{C}$ with Picard lattice isomorphic to that of S/K has infinite automorphism group.

This is done by taking a K3 surface S of degree 6 in $\mathbb{P}^4$ with two conjugate disjoint conics, such that these three curves generate the Picard group.

One shows that $\operatorname{Aut} \operatorname{Pic} S$ has a finite-index subgroup isomorphic to $\mathbb{Z}$ and that the product of the reflections in the two conics has infinite order. Therefore it generates a subgroup of finite index.

Example 3 (1)

As a particular case of the conjecture on the first slide, one would like to know about the surfaces $S_c : x^4 - y^4 = c(z^4 - w^4)$ for $c \in \mathbb{Q}$. They have obvious rational points $(\pm a : a : \pm 1 : 1)$, so the problem is to prove that rational points are Zariski dense.

In this conference, we have heard a lot about the parity conjecture, which would make short work of this problem. However, we want an unconditional result.

We do not know how to do this. However, we have shown that $\mathrm{Aut}\, S_c$ is finite for generic c (for which $[\mathbb{Q}(\zeta_8, \sqrt[4]{c}) : \mathbb{Q}] = 16$).

Example 3 (2)

It turns out that $\mathrm{Pic}\, S_c$ is isomorphic to a sublattice of index 4 in the lattice Λ of example 1 with Gram matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus -2I_4,$$

This allows us to determine a finite-index subgroup of $O(\mathrm{Pic}\, S_c)$.

(If Λ, Λ' are commensurable, then $O(\Lambda) \cap O(\Lambda')$ has finite index in both.)

Example 3 (3)

We check that there are 14 Galois orbits of lines and 8 of conics that give finite Coxeter groups and hence elements of R_{S_c}.

Writing these as matrices, we search for relations among the generators of our finite-index subgroup of $O(\mathrm{Pic}\, S_c)$ and these elements.

Then $\mathrm{Aut}\, S_c$ has a subgroup of finite index of the group with these generators and relations.

Magma shows that the group is finite.

End

Thank you for your attention.

Are there any questions?