# The number of integer points close to a polynomial

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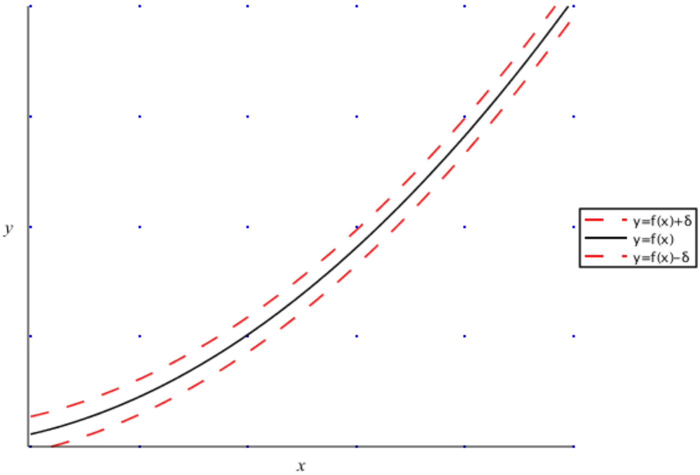
How large can $\mathcal{S}$ be for a given $f$?


y
x
y=f(x)+δ
y=f(x)
y=f(x)−δ

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- From Konyagin and Konyagin & Steger, we know that

$$\mathcal{S} \ll n \frac{X}{q^{1/n}} + n^{\omega(q)}.$$

- This is mostly best possible as well.

# A first general result

**Theorem**

*We have*

$$\mathcal{S} \ll_n \delta^{\frac{2}{n(n+1)}} X + \mathcal{R}$$

*where* $\mathcal{R}$ *is the maximal number of integer points in* $\Gamma_\delta$ *that are all on a polynomial of degree at most* $n$.

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Here and throughout the presentation, a lot of ideas are attributable to Filaseta, Huxley, Konyagin, Sargos, Steger, Swinnnerton-Dyer and Trifonov.

## Corollary

*Assume that the inequality*

$$\left|\alpha_n - \frac{r}{s}\right| \leq \frac{1}{s^2}$$

*holds for some integers* $r \in \mathbb{Z}$ *and* $s \in [1, X^n]$ *with gcd*$(r, s) = 1$*. Then,*

$$\mathcal{S} \ll_{n,\epsilon} \delta^{\frac{2}{n(n+1)}} X + \frac{X}{s^{1/n}} + X^{\epsilon}$$

*for each* $\epsilon > 0$*. For* $n = 1$ *the third term can be replaced by 1.*

# Sketch of the proof of the theorem

**Lemma**

*Let* $M_1 = (x_1, y_1), \ldots, M_{n+2} = (x_{n+2}, y_{n+2}) \in \Gamma_\delta \cap \mathbb{Z}^2$ *be ordered points according to* $x_1 < \cdots < x_{n+2}$*. Set*

$$\Lambda(M_1, \ldots, M_{n+2}) := \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n & y_1 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 1 & x_{n+2} & x_{n+2}^2 & \cdots & x_{n+2}^n & y_{n+2} \end{vmatrix}.$$

*Then, there are two possibilities:*

**(i)** $\Lambda(M_1, \ldots, M_{n+2}) \neq 0$ *in which case*
$$|x_{n+2} - x_1| \geq \left(\frac{1}{(n+2)\delta}\right)^{\frac{2}{n(n+1)}},$$

**(ii)** $\Lambda(M_1, \ldots, M_{n+2}) = 0$ *in which case all the points are on a polynomial curve*
$$\mathcal{C} := \{(x, y) \in \mathbb{R}^2 : \ y = P(x),\ \deg P(x) \leq n,\ P(x) \in \mathbb{Q}[x]\}.$$

- We order the points $(x, y) \in \Gamma_\delta \cap \mathbb{Z}^2$ according to the variable $x$.

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- We apply the previous lemma to groups of $(n+2)$ consecutive points.
- We end up with a contribution of $\ll \delta^{\frac{2}{n(n+1)}} X$ and a sequence of polyomials that contains many points of $\Gamma_\delta \cap \mathbb{Z}^2$.

- Fix a polynomial $Q(x) = \frac{P(x)}{q}$ with $P(x) = a_n x^n + \cdots + a_0 \in \mathbb{Z}[x]$ with $gcd(a_n, \ldots, a_0, q) = 1$. Assume that $(x_0, y_0) \in \Gamma_\delta \cap \mathbb{Z}^2$ and that $q \nmid P(x_0)$. Then,

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$$|f(x_0) - Q(x_0)| \geq \frac{1}{q} - \delta.$$

- We deduce that two consecutive sets of points must be far apart if $q \ll \frac{1}{\delta}$.
- In any case, the total contribution can be shown to be $\ll \delta^{\frac{2}{n(n+1)}} X$ with at most $\ll 1$ exceptions so that the result

$$\mathcal{S} \ll_n \delta^{\frac{2}{n(n+1)}} X + \mathcal{R}$$

holds.

# Sketch of the proof of the corollary

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- We distinguish 3 separated cases:

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\begin{aligned}
&(1) \quad q \gg \frac{1}{\delta},\\
&(2) \quad q \ll \frac{1}{\delta} \text{ and } |f(x) - P(x)| \ll \frac{1}{q} \text{ for each } x \in [X, 2X],\\
&(3) \quad q \ll \frac{1}{\delta} \text{ and } |f(z) - P(z)| \gg \frac{1}{q} \text{ for a } z \in [X, 2X].
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- The conclusion follows quite easily.

By considering $n+1$-tuples of integer points in $\Gamma_\delta \cap \mathbb{Z}^2$, we establish the combinatorial inequality

$$\mathcal{S} \ll_{n,\epsilon} \frac{X}{A} + \sum_{a_1,\ldots,a_n=1}^{A} t(a_1,\ldots,a_n) + \mathcal{M}_\epsilon$$

where $t(a_1,\ldots,a_n)$ counts the number of points $x$ with $(x,y) \in \Gamma_\delta \cap \mathbb{Z}^2$ such that

$$(x+a_1,y_1),\ (x+a_1+a_2,y_2),\ldots,\ (x+a_1+\cdots+a_n,y_n) \in \Gamma_\delta \cap \mathbb{Z}^2$$

with not all (the $n+1$) of them on a polynomial of degree at most $n-1$. $\mathcal{M}_\epsilon$ counts the contribution of sequences of ordered points in $\Gamma_\delta \cap \mathbb{Z}^2$ that are all on a polynomial of degree at most $n-1$.

It can be shown, as previously, that

$$\mathcal{M}_\epsilon \ll_{n,\epsilon,\epsilon_0} \delta^{\frac{1}{n-1}-\epsilon} X + \frac{X}{q^{\frac{1}{n-1}}} + X^{\epsilon_0}$$

where $q$ is the denominator of some rational approximation of the leading term $\alpha_n$.

From there, the idea is to fix $(a_1, \ldots, a_n)$ and to consider two consecutive points in $x, x+b \in t(a_1, \ldots, a_n)$.

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We write

$$d_0 := 0,\ d_1 := a_1,\ d_2 := a_1 + a_2, \ldots,\ d_n := a_1 + \cdots + a_n,$$

$$D_k := \prod_{\substack{0 \le i < j \le n \\ i,j \ne k}} (d_j - d_i), \quad D_{k,l} := \prod_{\substack{0 \le i < j \le n \\ i,j \ne k,l}} (d_j - d_i),$$

$$e := gcd(D_0, \ldots, D_n),$$

$$D_s := \max_j D_j, \quad D_{s,t} := \max_{j \ne s} D_{s,j}.$$

Some well known arguments lead us to the identity

$$nbe\alpha_n = c + 2n\theta \frac{\delta e D_{s,t}}{D_s}$$

where $b$ is considered as the variable that makes the term $nbe\alpha_n$ close to an integer $c$.

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We are thus led to estimate sums over $(a_1, \ldots, a_n)$ that contains $e$:

$$\sum_{a_1,\ldots,a_n=1}^{A} gcd(e,q), \quad \sum_{a_1,\ldots,a_n=1}^{A} e \text{ and } \sum_{a_1,\ldots,a_n=1}^{A} \frac{eD_{s,t}}{D_s}$$

for example.

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So far, we have

$$\mathcal{S} \ll_{n,\epsilon} X^{\epsilon}\left(\delta^{\beta_n} X + \frac{X}{q^{\frac{2}{n^2-n+2}}} + X^{1-\frac{2}{n^2+n}}\right)$$

when $n \geq 3$, $\left|\alpha_n - \frac{a}{q}\right| \leq \frac{1}{qX}$ ($q \leq X$ and $gcd(a,q) = 1$) and

| $n$ | 3 | 4 | 5 | 6 | $\geq 7$ |
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| $\beta_n$ | $\frac{5}{23}$ | $\frac{7}{50}$ | $\frac{3}{32}$ | $\frac{3}{46}$ | $\frac{2}{n^2-n}$ |

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For $n = 2$ and the same assumptions on $q$, we find

$$\mathcal{S} \ll_{\epsilon} X^{\epsilon}\left(\delta^{\frac{1}{2}}X + \frac{X}{q^{\frac{1}{3}}}\right).$$

Thank you!