Images of GL_2-type Galois representations

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First Example of a Galois Representation

- $E/\mathbb{Q}$ elliptic curve
- p a rational prime (fixed throughout talk)
- $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the group

$$E[p^n] = \{P \in E(\overline{\mathbb{Q}}) : p^n P = 0\} \cong (\mathbb{Z}/p^n\mathbb{Z})^2$$

- Get $\rho_{E,p^n} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$ that is unramified outside a finite set S, and for all $\ell \notin S$

$$\mathrm{tr}\, \rho_{E,p^n}(\mathrm{Frob}_\ell) \equiv 1 + \ell - \#E(\mathbb{F}_\ell) \bmod p^n.$$

- Get $\rho_{E,p^\infty} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Z}_p)$ such that for all $\ell \notin S$,

$$\mathrm{tr}\, \rho_{E,p^\infty}(\mathrm{Frob}_\ell) = 1 + \ell - \#E(\mathbb{F}_\ell).$$

Sources of more Galois representations

A: complete local noetherian pro-p ring
$\rho : \Pi \to \mathrm{GL}_2(A)$

Source	Π	A finite over
classical eigenforms	$G_{\mathbb{Q}}$	$\mathbb{Z}_p$
p-adic families of modular forms (Hida/Coleman families)	$G_{\mathbb{Q}}$	$\mathbb{Z}_p[[T]]$
Hilbert modular forms	G_F, F tot. real	$\mathbb{Z}_p$
p-adic families of HMFs	G_F	$\mathbb{Z}_p[[T_1, \ldots, T_n]]$
universal deformation	$\dim_{\mathbb{F}_p} \mathrm{Hom}(\Pi, \mathbb{F}_p) < \infty$	???

The Main Question

Question

If $\rho : \Pi \to \mathrm{GL}_2(A)$ is any of the above representations, what is the image of ρ?

Heuristic

The image of $\rho : \Pi \to \mathrm{GL}_2(A)$ should be as large as the symmetries of ρ allow.

What is a symmetry of ρ?

Henceforth, assume A is topologically generated by $\mathrm{tr}(\rho(\Pi))$ as a ring.

Definition

A **conjugate self-twist (CST)** of ρ is a pair (σ, η_σ), where $\sigma \in \mathrm{Aut}\, A$ and $\eta_\sigma : \Pi \to A^\times$ is a group homomorphism such that

$$\sigma(\mathrm{tr}\,\rho(g)) = \eta_\sigma(g)\,\mathrm{tr}\,\rho(g)$$

for all $g \in \Pi$. If (id_A, η) is a CST with $\eta \neq 1$, then ρ is **dihedral**.

$\Sigma_\rho = \{\sigma \in \mathrm{Aut}\, A : \exists \eta : \Pi \to A^\times, (\sigma, \eta)$ is a CST for $\rho\}$

A more precise expectation

Definition

Let A_0 be a ring and $0 \neq \mathfrak{a}_0 \subset A_0$ an ideal.

$$\Gamma_{A_0}(\mathfrak{a}_0) := \{x \in \mathrm{SL}_2(A_0) : x \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \bmod \mathfrak{a}_0\}.$$

We say ρ is A_0**-full** if (up to conjugation) $\mathrm{Im}\,\rho \supseteq \Gamma_{A_0}(\mathfrak{a}_0)$ for some $\mathfrak{a}_0 \neq 0$.

Heuristic

Let $\rho : \Pi \to \mathrm{GL}_2(A)$ *be irreducible, non-dihedral, and assume* $\mathrm{Im}\,\rho$ *is not finite. Then* ρ *should be* A^{Σ_ρ}*-full.*

An Example

- $f = \sum_{n=1}^{\infty} a_n q^n = q + \sqrt{44}q^2 + 3\sqrt{44}q^3 + 12q^4 + 132q^6 - 9\sqrt{44}q^7 - 20\sqrt{44}q^8 + 153q^9 + 252q^{11} + \ldots \in S_6(25)$
- Note that

$$a_n \in \begin{cases} \mathbb{Z} & n \equiv 1, 4 \bmod 5 \text{ (squares)} \\ \sqrt{44}\mathbb{Z} & n \equiv 2, 3 \bmod 5 \text{ (non-squares)} \\ \{0\} & n \equiv 0 \bmod 5. \end{cases}$$

- Taking (e.g.) $p = 11$, let $\sigma : \sqrt{44} \mapsto -\sqrt{44}$ and $\eta_\sigma = \left(\frac{\cdot}{5}\right)$.
- (σ, η_σ) is a CST for $\rho_{f,11}$: for all primes $\ell \nmid 55$ have

$$\begin{aligned} \sigma(\operatorname{tr} \rho_{f,11}(\mathrm{Frob}_\ell)) &= \sigma(a_\ell) = \eta_\sigma(\mathrm{Frob}_\ell) a_\ell \\ &= \eta_\sigma(\mathrm{Frob}_\ell) \operatorname{tr} \rho_{f,11}(\mathrm{Frob}_\ell). \end{aligned}$$

Why is A^{Σ_ρ} optimal? The Example Continued.

- Letting $\rho = \rho_{f,11}$:

$$\sigma(\mathrm{tr}\rho(g)) = \eta_\sigma(g)\,\mathrm{tr}\,\rho(g)$$

$\implies$ $\mathrm{tr}\,\rho(g)$ is an eigenvector for the $\mathbb{Q}_{11}$-linear map σ

$\implies$ $\mathrm{tr}\,\rho(g) \in \mathbb{Q}_{11} \cup \sqrt{44}\mathbb{Q}_{11}$

- If $\Gamma_{\mathbb{Z}_{11}[\sqrt{11}]}(\mathfrak{a}) \subset \mathrm{Im}\,\rho$ for some $0 \neq \mathfrak{a} \subset \mathbb{Z}_{11}[\sqrt{11}]$-ideal then

$$\begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix} \in \mathrm{Im}\,\rho$$

for all $a, b, c, d \in \mathfrak{a}$ such that $a + d + ad - bc = 0$.

- The Problem: $2 + a + d = \mathrm{tr}\begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix}$ doesn't stay in $\mathbb{Q}_{11} \cup \sqrt{44}\mathbb{Q}_{11}$ as a, d run over $\mathfrak{a}$.

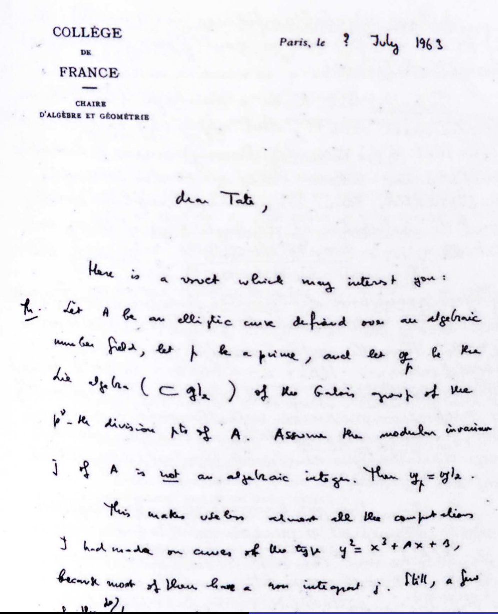
History

$\rho : \Pi \to \mathrm{GL}_2(A)$ irreducible, non-dihedral, infinite image arising from "Source" below

Source	Who proved ρ is A^{Σ_ρ}-full	Year
elliptic curves	Serre	1968
classical eigenforms	Ribet, Momose	1981
Hilbert modular forms	Nekovar	2012
Hida families	L.*	2016
Coleman families	Conti-Iovita-Tilouine*	2016

* = when $\bar{\rho} : \Pi \to \mathrm{GL}_2(A/\mathfrak{m}_A)$ is absolutely irreducible and satisfies a mild regularity condition and $p \neq 2$

The beginning of everything...



COLLÈGE DE FRANCE

CHAIRE D'ALGÈBRE ET GÉOMÉTRIE

Paris, le ? July 1963

dear Tate,

Here is a result which may interest you:

Th. Let A be an elliptic curve defined over an algebraic number field, let p be a prime, and let $\mathfrak{g}_p$ be the Lie algebra ($\subset \mathfrak{gl}_2$) of the Galois group of the p^ν-th division pts of A. Assume the modular invariant j of A is not an algebraic integer. Then $\mathfrak{g}_p = \mathfrak{gl}_2$.

This makes useless almost all the computations I had made on curves of the type $y^2 = x^3 + Ax + B$, because most of them have a non integral j. Still, a few

Our Result

$s : \mathbb{F}^\times \to A^\times$ Teichmuller lift

Definition

We say that ρ has **constant determinant** if $\det \rho = s(\det \bar{\rho})$, where $\bar{\rho} : \Pi \to \mathrm{GL}_2(\mathbb{F})$ is the mod $\mathfrak{m}_A$-reduction of ρ.

Definition

Say $\bar{\rho}$ is **regular** if $\exists \left(\begin{smallmatrix} \lambda_0 & 0 \\ 0 & \mu_0 \end{smallmatrix}\right) \in \mathrm{Im}\, \bar{\rho}$ such that $\pm 1 \neq \lambda_0 \mu_0^{-1} \in \mathbb{F}^{\Sigma_{\bar{\rho}}}$.

Theorem (Conti-L.-Medvedovsky, 2018)

Assume $\bar{\rho}$ is regular and ρ has constant determinant of 2-power order. If A is a domain with $p \neq 2$ and ρ is irreducible, non-dihedral, and $\mathrm{Im}\, \rho$ is not finite, then ρ is A^{Σ_ρ}-full.

Final Remarks

Theorem (Conti-L.-Medvedovsky, 2018)

Assume $\bar{\rho}$ is regular and ρ has constant determinant of 2-power order. If A is a domain with $p \neq 2$ and ρ is irreducible, non-dihedral, and $\mathrm{Im}\,\rho$ is not finite, then ρ is A^{Σ_ρ}-full.

- The constant determinant condition can always be achieved by twisting away the pro-p part of the determinant. Furthermore, once can deduce fullness results for the original representation from the twisted representation.
- Our work was inspired by work of Bellaïche. His work showed that ρ as in the theorem was A_0-full for a fairly mysterious ring A_0. The point of our work is to improve his A_0 and interpret this improved version in terms of conjugate self-twists of ρ, thus obtaining an optimality result.

Thank you!