Parity of ranks of abelian surfaces

Vladimir Dokchitser, joint work with Céline Maistret

July 10, 2018

Parity theorem

A/K (principally polarized) abelian variety over a number field.

Mordell-Weil Theorem

$$A(K) \simeq \mathbb{Z}^{rk_{A/K}} \oplus A(K)_{tors}.$$

Theorem 1 (DM)

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Parity conjecture

Hasse-Weil conjecture and functional equation

$L(A/K,s)$ has an analytic continuation to $\mathbb{C}$ and satisfies

$$L^*(A/K,s) = w_{A/K}\ L^*(A/K,2-s),$$

with $w_{A/K} = \prod_v w_v$ and $w_v(A/K_v) \in \{\pm 1\}$ the *local root numbers*.

Birch and Swinnerton-Dyer conjecture

$$ord_{s=1} L(A/K,s) = rk_{A/K}.$$

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$$rk_{A/K} \equiv \sum_v r_v(A/K_v) \mod 2, \qquad (-1)^{r_v} = w_v.$$

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Parity examples

Example : $E/\mathbb{Q} : y^2 + xy = x^3 - x, \quad \Delta_E = 5 \cdot 13.$

E has good reduction at $p \neq 5, 13 \implies r_p = 0$ for $p \neq 5, 13$.

E has split multiplicative reduction at $p = 5, 13 \implies r_5 = r_{13} = 1$.

Parity Conj. $\implies rk_{E/\mathbb{Q}} \equiv r_5 + r_{13} + r_\infty = 1 \mod 2. \implies E(\mathbb{Q})$ infinite.

Parity conjecture consequence 1:

Parity of $rk_{A_d/\mathbb{Q}}$ for quadratic twists A_d depend on $d \mod N$ for some N.
If dim A is odd, half the twists have even rank, and half odd.

Parity conjecture consequence 2:

All $A/\mathbb{Q}$ have even rank over $\mathbb{Q}(i, \sqrt{17})$.

Parity conjecture consequence 3:

$E : y^2 + y = x^3 + x^2 + x$ has positive rank over $\mathbb{Q}(\sqrt[3]{m})$ for all m.

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Parity theorem II

Theorem 2 (DM)

There is an invariant $r' \in \{0,1\}$ for pp abelian surfaces over local fields, such that for all pp abelian surfaces over number fields A/K

$$rk_{A/K} \equiv \sum_v r'(A/K_v) \mod 2,$$

provided $Ш_{A/F}$ is finite for $F = K(A[2])$.

Sketch of proof

Controlling rank when A admits an isogeny

Theorem (Cassels–Tate)

Assuming $Ш_{A/K}$ finite, if $\Phi : A \to A'$ is an isogeny of ppAVs, then

$$\frac{|Ш_{A/K}| \cdot Reg_{A/K} \cdot \prod_v c_v(A/K)}{|A(K)_{tors}|^2} = \frac{|Ш_{A'/K}| \cdot Reg_{A'/K} \cdot \prod_v c_v(A'/K)}{|A'(K)_{tors}|^2}.$$

Example: $E/\mathbb{Q} : y^2 + xy = x^3 - x, \qquad \Delta_E = 5 \cdot 13$, admits a 2-isogeny.

$c_5 = c_{13} = 1, \quad c'_5 = c'_{13} = 2, \quad c_\infty = 2c'_\infty.$

$\Rightarrow \frac{Reg_E}{Reg_{E'}} = \frac{|Ш(E)||E'(\mathbb{Q})_{tors}|^2 \prod c_v}{|Ш(E')||E(\mathbb{Q})_{tors}|^2 \prod c'_v} = \frac{\prod c_v}{\prod c'_v} \cdot \square = \frac{2}{4} \cdot \square \neq 1.$

$\Rightarrow E$ has a point of infinite order.

In fact, we can deduce that the rank is odd:

Corollary

If Φ is an isogeny of degree p such that $\Phi^*\Phi = [p]$ then $\frac{Reg_{A/K}}{Reg_{A'/K}} = p^{rk_{A/K}} \cdot \square$.

In particular, $rk_{A/K} = ord_p \frac{\prod_v c_v}{\prod_v c'_v} + ord_p \frac{|Ш_{A/K}[p^\infty]|}{|Ш_{A'/K}[p^\infty]|} \mod 2.$

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Reduction to the case of an isogeny with $\Phi^*\Phi = [2]$

Theorem (D²)

Let A/K be a semistable ppAV over a number field.
Let F/K be a finite Galois extension with $Ш_{A/F}$ finite, $G = Gal(F/K)$.
If Parity Conj. holds for A/F^H for all $H \leq Syl_2 G$, then it holds for A/K.

Corollary (D²)

Suppose A/K is semistable with $Ш_{A/F}$ finite for $F = K(A[2])$.
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Types of principally polarized abelian surfaces

Theorem (see Gonzales-Guàrdia-Rotger)

Let A/K be a pp abelian surface defined over K. Then either

- $A \simeq E_1 \times E_2$,
- $A \simeq Res_{F/K} E$, *or*
- $A \simeq Jac(C)$, *where C/K is a smooth curve of genus 2.*

In the first two cases, the parity theorem follows from analogous results for elliptic curves.

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Ш of Jacobians

Theorem (Poonen–Stoll)

Suppose $A = Jac(C)$ with $Ш_{A/K}$ is finite. Then $|Ш_{A/K}| = \square$ iff the number of places v with C/K_v deficient is even (and $2 \cdot \square$ otherwise).

Theorem

If $A = Jac(C)$, $Ш_{A/K}$ is finite and $\Phi : A \to Jac(C')$ with $\Phi^*\Phi = [2]$, then

$$rk_{A/K} = \sum_v ord_2 \frac{c_v}{c'_v} \frac{m_v}{m'_v} \mod 2,$$

with $m_v = 2$ if C is deficient at v and $m_v = 1$ otherwise.

Corollary (Theorem 2)

For pp abelian surfaces, if $Ш_{A/K(A[2])}$ is finite, then

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Problem: $r_v \neq r'_v$.

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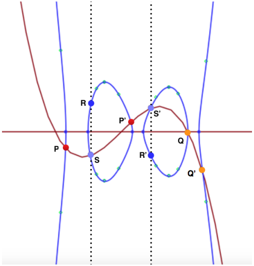
For pp abelian surfaces, if $Ш_{A/K(A[2])}$ is finite, then

$$rk_{A/K} = \sum_v r'_v \mod 2.$$

Problem: $r_v \neq r'_v$.

Arithmetic of the Jacobian of a genus 2 curve

$$C/K : y^2 = f(x), \quad deg(f) = 6.$$



Points on $Jac(C) \leftrightarrow [P, Q]$,
$P, Q \in C(\bar{K})$, Galois stable pair.

Adding points on $Jac(C)$:
Draw $y = cubic$ through P, P', Q, Q'.
$[P, P'] + [Q, Q'] + [S, S'] = 0$,
$-[S, S'] = [R, R']$.

2 torsion: $[T_i, T_k]$ where $T_i = (x_i, 0)$.

$Jac(C)$ admits an isogeny Φ with $\Phi\Phi^* = [2] \iff Gal(f) \leq C_2^3 \rtimes S_3$.

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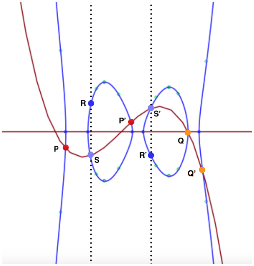

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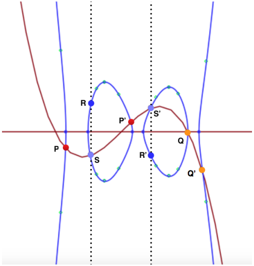

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Local discrepancy in terms of $f(x)$

For $C : y^2 = f(x)$ with deg $f(x) = 6$ and $Gal(f) \leq C_2 \times D_4$ $(=Syl_2 S_6)$, Maistret defined explicit $Gal(f)$-invariant polynomials l_{20}, l_{21}, l_{22}, l_{40}, l_{41}, l_{42}, l_{43}, l_{44}, l_{45}, l_{60}, l_{80}, ℓ in the roots of $f(x)$ and

Conjecture (Maistret)

$r_v = ord_2(\frac{c_v m_v}{c'_v m'_v}) + e_v \mod 2$, where

$$(-1)^{e_v} = (-1, l_{22} l_{41} l_{43} l_{60})(l_{20}, -l_{40} l_{44})(l_{40}, \ell l_{60} l_{43})(c, l_{23} l_{44} l_{80})(l_{23}, l_{41})$$

$$(l_{45}, -\ell l_{22} l_{21})(l_{44}, 2 l_{22} l_{42} l_{43})(l_{80}, -2 l_{41} l_{42} l_{60})(l_{42}, -l_{60} l_{43}),$$

is a product of Hilbert symbols at v.

By the product formula $\prod_v (-1)^{e_v} = 1$, so $\sum_v e_v$ is even.
Hence Maistret's Conjecture $\implies$ $rk_{A/K} = \sum r_v$ (Parity Conjecture), provided Ш is finite.

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Parity theorem

Theorem 3 (DM)

Suppose $C/K_v : y^2 = f(x)$ and $Gal(f) \leq C_2 \times D_4$. The Conjecture is true if either $v|\infty$, C is semistable and $v \nmid 2$, or if C is "lovely" and $v|2$.

Corollary (Theorem 1)

Let A/K be a pp semistable* abelian surface with $\text{Ш}_{A/K(A[2])}$. Then the Parity conjecture holds for A/K, that is $rk_{A/K} = \sum r_v \mod 2$.

Maistret's conjecture is a purely local statement.
Proof of 3: classify all reduction types of C/K_v and describe root numbers, Tamagawa numbers and deficiency in terms of the roots of $f(x)$.
(="Cluster" machinery for hyperelliptic curves by D^{2}M+Adam Morgan).

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Thank you for your attention