The j-function, the golden ratio, and rigid meromorphic cocycles

Henri Darmon, McGill University

CNTA XV, July 2018

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• It was at CNTA 0 that I really met for the first time many future colleagues: Claude, Jean-Marie, Damien, Hershy, Ram and Kumar Murty...

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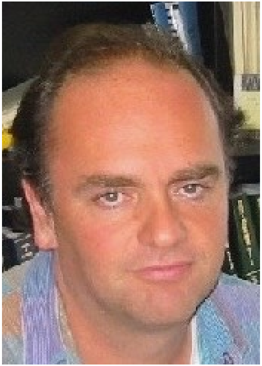

Andrew Today

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This is joint work with Jan Vonk

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It generates an interesting cubic extension of $\mathbb{Q}(\sqrt{-23})$.

Gauss composition and class groups

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Class groups have been intensely studied since the time of Gauss, and are fundamental objects in number theory.

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Theorem (Kronecker)

For quadratic $\tau \in \mathcal{H}$ of fundamental discriminant $D < 0$, the singular modulus $j(\tau)$ is an algebraic integer in H_D, and generates this field.

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expresses a "powerful" number as a sum of two "powerful" numbers.

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Similar examples arise from any $J(\tau_1, \tau_2)$ for which $j(\tau_1)$ and $j(\tau_2)$ are integers.

The abc conjecture

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Why doesn't complex multiplication contradict the abc conjecture?

The Gauss class number problem

Theorem (Gauss; Heilbronn; Baker, Heegner, Stark)

There are finitely many $D < 0$ with for which $G_D = \{1\}$, namely

$$-3, -4, -7, -8, -11, -12, -16, -19, -27, -28, -43 - 67, -163.$$

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Granville, Stark (2000): A strong form of the abc conjecture implies the non-existence of Siegel zeroes for L-functions attached to odd Dirichlet characters.

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One might disprove the abc conjecture by *extending* the definition of $J(\tau_1, \tau_2)$ so that it can be evaluated at real quadratic, and not just imaginary quadratic, arguments!

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The rest of the talk will try to explain this cryptic answer.

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The field $\mathbb{Q}_p$ plays the role of $\mathbb{R}$, while the counterpart of $\mathbb{C}$ is

$$\mathbb{C}_p := \widehat{\bar{\mathbb{Q}}}_p.$$

Drinfeld's p-adic upper-half plane

Likewise, the complex upper half plane $\mathcal{H}$ with its action of $\mathbf{SL}_2(\mathbb{R})$ by Möbius transformations, admits a good p-adic analogue.

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This is the proper generalisation of holomorphic functions on $\mathcal{H}$.

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Goal: Define a "convincing" p-adic avatar of j which can be meaningfully *evaluated* at RM points, leading to singular moduli for real quadratic $\tau \in \mathcal{H}_p$.

Rigid meromorphic functions on $\mathcal{H}_p$

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Let $\mathcal{M} :=$ the ring of rigid meromorphic functions on $\mathcal{H}_p$.

$\mathbf{SL}_2(\mathbb{Z})$-invariant functions

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This is a futile, because: The actions of $\mathbf{SL}_2(\mathbb{Z})$ on $\mathcal{H}_p$ is not discrete in the p-adic topology.

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This does nothing to alleviate the problems of the previous slide:

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Thesis: rigid meromorphic cocycles play the role of meromorphic modular functions in extending CM theory to real quadratic fields.

Evaluating a modular cocycle at an RM point

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$$J[\gamma\tau] = J[\tau], \qquad \text{for all } \gamma \in \Gamma.$$

Explicit rigid meromorphic cocycles

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$$J_\tau(S)(z) \sim \prod_{w \in \Sigma_\tau} (z - w)^{\mathrm{sgn}(w)}, \qquad S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

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If $\Delta = \sum_i n_i[\tau_i] \in \mathrm{Div}(\Gamma\backslash\mathcal{H}_p)$, write

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Fields of definition

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Definition

The *field of definition of J*, denoted H_J, is the compositum of H_D as D ranges over the discriminants of all the RM points in the support of the divisor of J.

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Proposition

If p is a monstrous prime and τ is any RM point on $\mathcal{H}_p$, then $(\tau) - (p\tau)$ is a principal divisor.

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The quantity $J_p(\tau_1, \tau_2)$ belongs to the compositum $H_1 H_2$ of the ring class fields of $K_1 = \mathbb{Q}(\tau_1)$ and $K_2 = \mathbb{Q}(\tau_2)$ attached to τ_1 and τ_2 respectively,

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$$J_\infty(\tau_1, \tau_2) := J(\tau_1, \tau_2) = j(\tau_1) - j(\tau_2)$$

of Gross-Zagier.

Gross-Zagier factorisations

Let τ_1 and $\tau_2 \in \mathcal{H}$ be CM points with (negative) discriminants D_1 and D_2, and let $\mathcal{O}_{D_1}$ and $\mathcal{O}_{D_2}$ be the associated quadratic orders.

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$$\operatorname{ord}_q J_\infty(\tau_1, \tau_2) = e_{q\infty}(\tau_1, \tau_2).$$

Factorisations of real quadratic singular moduli

For $\tau_1, \tau_2 \in \mathcal{H}_p^{\mathrm{RM}}$, one can likewise define an arithmetic quantity $e_{qp}(\tau_1, \tau_2)$ involving the **indefinite** quaternion algebra B_{qp} ramified at q and p.

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The Riemann surface $\Gamma_{pq} \backslash \mathcal{H}$ is called the *Shimura curve* X_{qp} attached to the pair (p, q).

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For $\tau_1, \tau_2 \in \mathcal{H}_p^{\mathrm{RM}}$, one can likewise define an arithmetic quantity $e_{qp}(\tau_1, \tau_2)$ involving the **indefinite** quaternion algebra B_{qp} ramified at q and p.

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$e_{qp}(\tau_1, \tau_2) :=$ weighted intersection of γ_1 and γ_2 on X_{qp}.

A Gross-Zagier-style factorisation

Let $\tau_1 \in \mathcal{H}_p^{D_1}$ and $\tau_2 \in \mathcal{H}_p^{D_2}$.

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$$\operatorname{ord}_q J_p(\tau_1, \tau_2) = e_{qp}(\tau_1, \tau_2).$$

An example

James Rickards has developed and implemented efficient algorithms for computing the q-weighted topological intersection numbers $e_{qp}(\tau_1, \tau_2)$ of real quadratic geodesics on the Shimura curve X_{qp}.

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James Rickards: The topological intersection $e_{q2}(\tau_1, \tau_2)$ is zero except for the primes q in the following table:

q	7	19	31	73	109	151	163	397	457	463
$e_{q2}(\tau_1, \tau_2)$	7	2	2	1	2	2	1	1	1	1

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Likewise there seem to be no applications to Siegel zeroes for even Dirichlet characters, à la Granville-Stark.

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The proof rests on a remarkable recent work of Samit Dasgupta and Mahesh Kakde, building on the proof of the "rank one p-adic Gross-Stark conjecture" by Dasgupta-Darmon-Pollack (2006).

Thank you for your attention.