Felder-Varchenko modular forms and the arithmetic of cubic number fields.

(on Eiseinstein's Jugendtraum)

Pierre Charollois,
Institut de Mathématiques de Jussieu-PRG, Université Paris 6.

CNTA XV, Université Laval, Québec, 13 juillet 2018.

Elliptic functions are meromorphic functions with poles located on a period lattice $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$, e.g.

$$E_k(\omega, z) = \sum_{a,b \in \mathbb{Z}^2} \frac{1}{(a\omega_1 + b\omega_2 + z)^k}.$$

Beyond $SL_2(\mathbb{Z})$**...** $SL_3(\mathbb{Z})$?

Obstruction : It's been long known that there are no meromorphic functions with 3 lin. independent periods. Period.

Elliptic functions are meromorphic functions with poles located on a period lattice $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$, e.g.

$$E_k(\omega, z) = \sum_{a,b \in \mathbb{Z}^2} \frac{1}{(a\omega_1 + b\omega_2 + z)^k}.$$

Beyond $SL_2(\mathbb{Z})$**...** $SL_3(\mathbb{Z})$?

Obstruction : It's been long known that there are no meromorphic functions with 3 lin. independent periods ... ?

21 years old G. Eisenstein (1844) observes :

$$\sum_{a,b,c\in\mathbb{Z}^3} \frac{1}{(a\omega_1 + b\omega_2 + c\omega_3 + z)^k} \text{ diverges.}$$

21 years old G. Eisenstein (1844) observes :

$$\sum_{a,b,c\in\mathbb{Z}^3} \frac{1}{(a\omega_1 + b\omega_2 + c\omega_3 + z)^k} \text{ diverges.}$$

He suggests :

A long quest for setting others than $SL_2(\mathbb{Z})$

**21 years old G. Eisenstein (1844)** observes :

$$\sum_{a,b,c\in\mathbb{Z}^3} \frac{1}{(a\omega_1 + b\omega_2 + c\omega_3 + z)^k} \text{ diverges.}$$ **He suggests** :

"**However no such inconvenience arises if we impose some restrictions on the summation indices such as inequalities conditions....**
... **There is a large class of such functions that is closely connected to Number Theory....** These functions possess very remarkable properties; they lead to the most beautiful researches, and seem to lie at the crossroads where the most difficult parts of analysis and number theory meet."

three remarks :

• Eisenstein actually emphasized on **infinite products,** considering infinite series as a secondary issue.

• Historical aside :
G. Eisenstein (1823-1852), and
L. Kronecker (1823-1891) were students and friends in Berlin.

• This 1844 paper of Eisenstein has been mocked by Jacobi, and subsequently has been largely ignored (quoted less than 10 times in the next 150 years). **It certainly deserves to be re-evaluated.**

3 remarks :

• Eisenstein actually emphasized on **infinite products,** considering infinite series as a secondary issue.

• Historical aside :
G. Eisenstein (1823-1852), and
L. Kronecker (1823-1891) were students and friends in Berlin.

• This 1844 paper of Eisenstein has been ridiculed by Jacobi, and subsequently has been largely ignored (quoted less than 10 times in the next 150 years). **It certainly deserves to be re-evaluated.**

• **Our aim :** armed with this predictive (and cryptic) assertion of Eisenstein, we want to revisit some intriguing identities discovered by Felder-Varchenko (2005) in connection with the KZB Heat equation, and to merge them with Number Theory.

Definition

(Felder-Varchenko, 2005). For $\tau, \sigma \in \mathcal{H} \times \mathcal{H}$, $k \geq 0$, $q = \mathbf{e}(\tau), r = \mathbf{e}(\sigma)$, they consider

$$F_k(\tau, \sigma) = (2\pi i)^k \sum_{n \geq 1} n^{k-1} \frac{q^n - (-1)^k r^n}{(1-q^n)(1-r^n)}$$
$$= \left(\sum_{a \in \mathbb{Z}, b, c > 0} - \sum_{a \in \mathbb{Z}, b, c < 0} + \text{bdary} \right) \frac{1}{(a + b\tau + c\sigma)^k},$$

where the last line (an observation of Zagier) only holds for odd $k \geq 5$. "bdary" means a contribution $a \in \mathbb{Z}, b = 0, 0 \neq c \in \mathbb{Z}$ and $b \leftrightarrow c$.

Definition

(Felder-Varchenko, 2005). For $\tau, \sigma \in \mathcal{H} \times \mathcal{H}$, $k \geq 0$, $q = \mathbf{e}(\tau), r = \mathbf{e}(\sigma)$, they consider

$$F_k(\tau,\sigma) = (2\pi i)^k \sum_{n\geq 1} n^{k-1} \frac{q^n - (-1)^k r^n}{(1-q^n)(1-r^n)}$$

$$= \left(\sum_{a\in\mathbb{Z}, b,c>0} - \sum_{a\in\mathbb{Z}, b,c<0} + \text{bdary} \right) \frac{1}{(a+b\tau+c\sigma)^k},$$

where the last line only holds for odd $k \geq 5$.

Theorem

(F.V.) If $\tau, \sigma, \frac{\sigma}{\tau} \in \mathcal{H}$, then

$$F_k(\tau,\sigma) - \tau^{-k} F_k(-\frac{1}{\tau}, \frac{\sigma}{\tau}) + (-\sigma)^{-k} F_k(-\frac{\tau}{\sigma}, -\frac{1}{\sigma}) = \begin{cases} 0 & \textit{if } k \geq 4, \\ i\pi P_k \in i\pi\mathbb{Q}(\tau,\sigma), & k = 1,2,3 \end{cases}$$

Theorem

(F.V.) If $\tau, \sigma, \frac{\sigma}{\tau} \in \mathcal{H}$, *then*

$$F_k(\tau,\sigma) - \tau^{-k} F_k(-\frac{1}{\tau}, \frac{\sigma}{\tau}) + (-\sigma)^{-k} F_k(-\frac{\tau}{\sigma}, -\frac{1}{\sigma}) = \begin{cases} 0 & if\ \ k \geq 4, \\ i\pi P_k \in i\pi\mathbb{Q}(\tau,\sigma), k = 1,2,3 \end{cases}$$

What is going on here ?

Theorem

(F.V.) If $\tau, \sigma, \frac{\sigma}{\tau} \in \mathcal{H}$*, then*

$$F_k(\tau,\sigma) - \tau^{-k} F_k(-\frac{1}{\tau}, \frac{\sigma}{\tau}) + (-\sigma)^{-k} F_k(-\frac{\tau}{\sigma}, -\frac{1}{\sigma}) = \begin{cases} 0 & \textit{if } k \geq 4, \\ i\pi P_k \in i\pi\mathbb{Q}(\tau,\sigma), k = 1, 2, 3 \end{cases}$$

This identity is reminiscent of the more familiar (k even)

$$E_k(\tau) := 2\zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} n^{k-1} \frac{q^n}{(1-q^n)}$$

$$= \text{boundary} + \sum_{a \in \mathbb{Z}, b > 0} \frac{1}{(a + b\tau)^k},$$

which satisfies

$$E_k(\tau) - \tau^{-k} E_k(-\frac{1}{\tau}) = \begin{cases} 0 & \text{if } k \geq 4 \text{ (i.e. Eisenstein series are modular)}, \\ i\pi P_k, P_k \in \mathbb{Q}(\tau) & \text{if the weight is 2.} \end{cases}$$

An improved Felder-Varchenko identity

For $\tau, \sigma, \frac{\sigma}{\tau} \in \mathcal{H},\ F_k(\tau,\sigma) = cst. \sum_{n\geq 1} n^{k-1} \frac{q^n - (-1)^k r^n}{(1-q^n)(1-r^n)}$:

$$F_k(\tau,\sigma) - \tau^{-k} F_k(-\frac{1}{\tau}, \frac{\sigma}{\tau}) + (-\sigma)^{-k} F_k(-\frac{\tau}{\sigma}, -\frac{1}{\sigma}) = \begin{cases} 0 \ \text{ if } \ k \geq 4, \\ i\pi P_k \in i\pi\mathbb{Q}(\tau,\sigma), k = 1,2,3 \end{cases}$$

while
$E_k(\tau) = cst. \sum_{n\geq 1} n^{k-1} \frac{q^n}{(1-q^n)}$ satisfies

$$E_k(\tau) - \tau^{-k} E_k(-\frac{1}{\tau}) = \begin{cases} 0 \ \text{ if } \ k \geq 4 \ , \\ \ i\pi P_k, P_k \in \mathbb{Q}(\tau) \text{ if } \ k = 2. \end{cases}$$

An improved Felder-Varchenko identity

For $\tau, \sigma, \frac{\sigma}{\tau} \in \mathcal{H},\ F_k(\tau,\sigma) = cst. \sum_{n\geq 1} n^{k-1} \frac{q^n - (-1)^k r^n}{(1-q^n)(1-r^n)}$:

$$F_k(\tau,\sigma) - \tau^{-k} F_k(-\frac{1}{\tau}, \frac{\sigma}{\tau}) + (-\sigma)^{-k} F_k(-\frac{\tau}{\sigma}, -\frac{1}{\sigma}) = \begin{cases} 0 \ \text{ if } \ k \geq 4, \\ i\pi P_k \in i\pi\mathbb{Q}(\tau,\sigma), k = 1, 2, 3 \end{cases}$$

while $E_k(\tau) = cst. \sum_{n\geq 1} n^{k-1} \frac{q^n}{(1-q^n)}$ satisfies

$$E_k(\tau) - \tau^{-k} E_k(-\frac{1}{\tau}) = \begin{cases} 0 \ \text{ if } k \geq 4 \text{ (modular)}, \\ i\pi P_k, P_k \in \mathbb{Q}(\tau) \text{ if } k = 2 \text{ (almost modular)}. \end{cases}$$

An improved Felder-Varchenko identity

For $\tau, \sigma, \frac{\sigma}{\tau} \in \mathcal{H},\ F_k(\tau,\sigma) = cst. \sum_{n\geq 1} n^{k-1} \frac{q^n-(-1)^k r^n}{(1-q^n)(1-r^n)}$:

$$F_k(\tau,\sigma) - \tau^{-k} F_k(-\frac{1}{\tau}, \frac{\sigma}{\tau}) + (-\sigma)^{-k} F_k(-\frac{\tau}{\sigma}, -\frac{1}{\sigma}) = \begin{cases} 0 \ \text{ if } \ k \geq 4, \\ i\pi P_k \in i\pi\mathbb{Q}(\tau,\sigma), k=1,2,3 \end{cases}$$

while $E_k(\tau) = cst. \sum_{n\geq 1} n^{k-1} \frac{q^n}{(1-q^n)}$ satisfies

$$E_k(\tau) - \tau^{-k} E_k(-\frac{1}{\tau}) = \begin{cases} 0 \ \text{ if } k \geq 4 \text{ (modular)}, \\ i\pi P_k, P_k \in \mathbb{Q}(\tau) \text{ if } k = 2 \text{ (almost modular)}. \end{cases}$$

key insight : go further down in the weight parameter (and keep your eyes open for help from Great Masters of the past).

An improved Felder-Varchenko identity

For $\tau, \sigma, \frac{\sigma}{\tau} \in \mathcal{H},\ F_k(\tau,\sigma) = cst. \sum_{n\geq 1} n^{k-1} \frac{q^n-(-1)^k r^n}{(1-q^n)(1-r^n)}$:

$$F_k(\tau,\sigma) - \tau^{-k} F_k(-\frac{1}{\tau}, \frac{\sigma}{\tau}) + (-\sigma)^{-k} F_k(-\frac{\tau}{\sigma}, -\frac{1}{\sigma}) = \begin{cases} 0 \text{ if } k \geq 4, \\ i\pi P_k \in i\pi\mathbb{Q}(\tau,\sigma), k=1,2,3 \end{cases}$$

while $E_k(\tau) = cst. \sum_{n\geq 1} n^{k-1} \frac{q^n}{(1-q^n)}$ satisfies

$$E_k(\tau) - \tau^{-k} E_k(-\frac{1}{\tau}) = \begin{cases} 0 \text{ if } k \geq 4 \text{ (modular)}, \\ i\pi P_k, P_k \in \mathbb{Q}(\tau) \text{ if } k = 2 \text{ (almost modular)}, \\ i\pi P_k, P_k \in \mathbb{Q}(\tau, \log\tau) \text{ if } k = 0 \text{ (Dedekind's } \log\eta), \end{cases}$$

An improved Felder-Varchenko identity

For $\tau, \sigma, \frac{\sigma}{\tau} \in \mathcal{H},\ F_k(\tau,\sigma) = cst. \sum_{n\geq 1} n^{k-1} \frac{q^n - (-1)^k r^n}{(1-q^n)(1-r^n)}$:

$$F_k(\tau,\sigma) - \tau^{-k} F_k(-\frac{1}{\tau}, \frac{\sigma}{\tau}) + (-\sigma)^{-k} F_k(-\frac{\tau}{\sigma}, -\frac{1}{\sigma}) = \begin{cases} 0 \text{ if } k \geq 4, \\ i\pi P_k \in i\pi\mathbb{Q}(\tau,\sigma), k = 1,2,3 \end{cases}$$

while $E_k(\tau) = cst. \sum_{n\geq 1} n^{k-1} \frac{q^n}{(1-q^n)}$ satisfies

$$E_k(\tau) - \tau^{-k} E_k(-\frac{1}{\tau}) = \begin{cases} 0 \text{ if } k \geq 4 \text{ (modular)}, \\ i\pi P_k, P_k \in \mathbb{Q}(\tau) \text{ if } k = 2 \text{ (almost modular)} \\ i\pi P_k, P_k \in \mathbb{Q}(\tau, \log\tau) \text{ if } k = 0 \text{ (Dedekind's } \log\eta), \\ i\pi P_k, P_k \in \mathbb{Q}(\tau) \text{ if } k < 0 \text{ (Lerch, Ramanujan)}. \end{cases}$$

Theorem

(P.C.)

For $\tau, \sigma, \frac{\sigma}{\tau} \in \mathcal{H},\ F_k(\tau,\sigma) = cst. \sum_{n\geq 1} n^{k-1} \frac{q^n - (-1)^k r^n}{(1-q^n)(1-r^n)}$ *:*

$$F_k(\tau,\sigma) - \tau^{-k} F_k(-\frac{1}{\tau}, \frac{\sigma}{\tau}) + (-\sigma)^{-k} F_k(-\frac{\tau}{\sigma}, -\frac{1}{\sigma}) = \begin{cases} 0 \quad \mathit{if} \ k \geq 4, \\ i\pi P_k \in i\pi\mathbb{Q}(\tau,\sigma), k = 1, 2, 3 \\ i\pi P_k \in i\pi\mathbb{Q}(\tau,\sigma), k \leq 0. \end{cases}$$

while $E_k(\tau) = cst. \sum_{n\geq 1} n^{k-1} \frac{q^n}{(1-q^n)}$ satisfies

$$E_k(\tau) - \tau^{-k} E_k(-\frac{1}{\tau}) = \begin{cases} 0 \ \text{ if } k \geq 4 \text{ (modular)}, \\ i\pi P_k, P_k \in \mathbb{Q}(\tau) \text{ if } k = 2 \text{ (almost modular)} \\ i\pi P_k, P_k \in \mathbb{Q}(\tau, \log\tau) \text{ if } k = 0 \text{ (Dedekind's } \log\eta\text{)}, \\ i\pi P_k, P_k \in \mathbb{Q}(\tau) \text{ if } k < 0 \text{ (Lerch, Ramanujan)}. \end{cases}$$

Interpretation as cocycle relations for $SL_2(\mathbb{Z}), SL_3(\mathbb{Z})$.

Rewrite

$$F_k(\tau,\sigma)-\tau^{-k}F_k(-\frac{1}{\tau},\frac{\sigma}{\tau})+(-\sigma)^{-k}F_k(-\frac{\tau}{\sigma},-\frac{1}{\sigma}) = i\pi P_k \in i\pi\mathbb{Q}(\tau,\sigma), k \leq 0$$

as

$$F^k_{e_1,e_2}(\tau,\sigma) - F^k_{e_1,e_3}(\tau,\sigma) + F^k_{e_2,e_3}(\tau,\sigma) = i\pi P^k_{e_1,e_2,e_3}(\tau,\sigma) \in i\pi\mathbb{Q}(\tau,\sigma),$$

where $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$ stand for the canonical basis of $\mathbb{Z}^3$,

while $E_k(\tau) = cst. \sum_{n\geq 1} n^{k-1}\frac{q^n}{(1-q^n)}$ satisfies

$$E_k(\tau) - \tau^{-k}E_k(-\frac{1}{\tau}) = E_k - E_k \mid S = i\pi P^k(S)(\tau).$$

Interpretation as cocycle relations for $SL_2(\mathbb{Z}), SL_3(\mathbb{Z})$.

Rewrite

$$F_k(\tau,\sigma)-\tau^{-k}F_k(-\frac{1}{\tau},\frac{\sigma}{\tau})+(-\sigma)^{-k}F_k(-\frac{\tau}{\sigma},-\frac{1}{\sigma}) = i\pi P_k \in i\pi\mathbb{Q}(\tau,\sigma), k \leq 0.$$

as

$$F^k_{e_1,e_2}(\tau,\sigma) - F^k_{e_1,e_3}(\tau,\sigma) + F^k_{e_2,e_3}(\tau,\sigma) = i\pi P^k_{e_1,e_2,e_3}(\tau,\sigma) \in i\pi\mathbb{Q}(\tau,\sigma),$$

where $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$ stand for the canonical basis of $\mathbb{Z}^3$,

while $E_k(\tau) = cst. \sum_{n\geq 1} n^{k-1}\frac{q^n}{(1-q^n)}$ satisfies

$$E_k - E_k \mid S = i\pi P(S)(\tau),$$

and more generally

$$E_k \mid (\mathrm{Id} - A) = i\pi P(A)(\tau),\ \text{any}\, A \in SL_2(\mathbb{Z}).$$

Interpretation as cocycle relations for $SL_2(\mathbb{Z})$, $SL_3(\mathbb{Z})$.

$$F^k_{e_1,e_2}(\tau,\sigma) - F^k_{e_1,e_3}(\tau,\sigma) + F^k_{e_2,e_3}(\tau,\sigma) = i\pi P^k_{e_1,e_2,e_3}(\tau,\sigma) \in i\pi\mathbb{Q}(\tau,\sigma),$$

while

$$\boxed{E_k \mid (\mathrm{Id} - A) = i\pi P(A)(\tau) \in \mathbb{Q}(\tau),\ \text{any } A \in SL_2(\mathbb{Z}),}$$

which implies that the period " polynomials " satisfy

$$P(AB)(\tau) = P(A)(B\tau) + P(B)(\tau).$$

Consequence :
$A \mapsto P(A)(\tau)$ is a 1-cocycle for $SL_2(\mathbb{Z})$.

Interpretation as cocycle relations for $SL_2(\mathbb{Z}), SL_3(\mathbb{Z})$.

$$F^k_{e_1,e_2}(\tau,\sigma) - F^k_{e_1,e_3}(\tau,\sigma) + F^k_{e_2,e_3}(\tau,\sigma) = i\pi P^k_{e_1,e_2,e_3}(\tau,\sigma) \in i\pi\mathbb{Q}(\tau,\sigma),$$

while

$$\boxed{E_k \mid (\mathrm{Id} - A) = i\pi P(A)(\tau) \in \mathbb{Q}(\tau),\ \text{any}\, A \in SL_2(\mathbb{Z}),}$$

which implies that the period " polynomials " satisfy

$$P(AB)(\tau) = P(A)(B\tau) + P(B)(\tau).$$

Consequence :
$A \mapsto P(A)(\tau)$ is a 1-cocycle for $SL_2(\mathbb{Z})$ (and Lerch-Ramanujan provided a splitting for it).

Interpretation as cocycle relations for $SL_2(\mathbb{Z}), SL_3(\mathbb{Z})$.

$$F^k_{e_1,e_2}(\tau,\sigma) - F^k_{e_1,e_3}(\tau,\sigma) + F^k_{e_2,e_3}(\tau,\sigma) = i\pi P^k_{e_1,e_2,e_3}(\tau,\sigma) \in i\pi\mathbb{Q}(\tau,\sigma),$$

while

$$\boxed{E_k \mid (\mathrm{Id} - A) = i\pi P(A)(\tau) \in \mathbb{Q}(\tau),\ \text{any } A \in SL_2(\mathbb{Z}),}$$

Consequence :
$A \mapsto P(A)(\tau)$ is a 1-cocycle for $SL_2(\mathbb{Z})$ (and Lerch-Ramanujan provided a splitting for it).
The cohomology class attached to this 1-cocycle is non trivial : let $F/\mathbb{Q}$ be a real quadratic field. The fundamental unit ϵ_F acts on $\mathcal{O}_F = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$, providing us with a cycle A_F to pair with our 1-cocycle :

$$< P_k, A_F >= \zeta_F(1-k) \in \mathbb{Q}^\times. \ \text{(Meyer-Klingen-Siegel)}.$$

It is also a major tool for p-adic works : Darmon-Dasgupta, Chapdelaine.

Interpretation as cocycle relations for $SL_2(\mathbb{Z}), SL_3(\mathbb{Z})$.

$$F^k_{e_1,e_2}(\tau,\sigma) - F^k_{e_1,e_3}(\tau,\sigma) + F^k_{e_2,e_3}(\tau,\sigma) = i\pi P^k_{e_1,e_2,e_3}(\tau,\sigma) \in i\pi\mathbb{Q}(\tau,\sigma).$$

Theorem

(P.C.) there is a collection of homog. degree k merom. functions $F^k_{a,b}(x)$, defined for x in certain open subsets of $\mathbb{CP}^2 - \mathbb{RP}^2$ and indexed by pairs (a,b) of primitive vectors in $\mathbb{Z}^3$, that satisfy $\forall\, g \in SL_3(\mathbb{Z})$
$F^k_{ga,gb}(gx) = F^k_{a,b}(x)$, and

$$F^k_{a,b}(\tau,\sigma) - F^k_{a,c}(\tau,\sigma) + F^k_{b,c}(\tau,\sigma) = i\pi P^k_{a,b,c}(\tau,\sigma) \in i\pi\mathbb{Q}(\tau,\sigma).$$

---

reminder :

$$\boxed{E_k \mid (\mathrm{Id} - A) = i\pi P(A)(\tau) \in \mathbb{Q}(\tau),\ \text{any}\, A \in SL_2(\mathbb{Z}),}$$

(which can also be formally recasted as

$$E^k_{e_1} - E^k_{Ae_1} = i\pi P_{e_1,Ae_1}(\tau) \in \mathbb{Q}(\tau),\ \text{any}\, A \in SL_2(\mathbb{Z}).)$$

Our favourite 2-cocycle for $SL_3(\mathbb{Z})$.

Theorem

(P.C.)(copied from prev. slide) there is a collection of homog. degree k merom. functions $F^k_{a,b}(x)$, defined for x in certain open subsets of $\mathbb{CP}^2 - \mathbb{RP}^2$ and indexed by pairs (a, b) of primitive vectors in $\mathbb{Z}^3$, that satisfy $F^k_{ga,gb}(gx) = F^k_{a,b}(x)$, $\forall\, g \in SL_3(\mathbb{Z})$, and

$$F^k_{a,b}(\tau,\sigma) - F^k_{a,c}(\tau,\sigma) + F^k_{b,c}(\tau,\sigma) = i\pi P^k_{a,b,c}(\tau,\sigma) \in i\pi\mathbb{Q}(\tau,\sigma).$$

Consequence :

$(a, b, c) \mapsto P^k_{a,b,c} \in \mathbb{Q}(x_1, x_2, x_3)$ is a homog. 2-cocycle for $SL_3(\mathbb{Z})$.

- **Q** : What is this 2-cocycle ? Is it trivial ?

Our favourite 2-cocycle for $SL_3(\mathbb{Z})$.

$$F^k_{a,b}(\tau,\sigma) - F^k_{a,c}(\tau,\sigma) + F^k_{b,c}(\tau,\sigma) = i\pi P^k_{a,b,c}(\tau,\sigma) \in i\pi\mathbb{Q}(\tau,\sigma).$$

Consequence :
$(a,b,c) \mapsto P^k_{a,b,c} \in \mathbb{Q}(x_1,x_2,x_3)$ is a homog. 2-cocycle for $SL_3(\mathbb{Z})$.

- **Q** : What is this 2-cocycle ? Is it trivial ?

Theorem

(P.C.) This 2-cocycle on $SL_3(\mathbb{Z})$ is equal to the "Shintani-cocycle" recently introduced in C-Dasgupta-Greenberg following work of Solomon, as a variation of the Sczech-cocycle (1993). It is non-trivial because it parametrizes values of zeta functions of tot. real cubic fields.

Our favourite 2-cocycle for $SL_3(\mathbb{Z})$.

$$F^k_{a,b}(\tau,\sigma) - F^k_{a,c}(\tau,\sigma) + F^k_{b,c}(\tau,\sigma) = i\pi P^k_{a,b,c}(\tau,\sigma) \in i\pi\mathbb{Q}(\tau,\sigma).$$

Consequence :
$(a,b,c) \mapsto P^k_{a,b,c} \in \mathbb{Q}(x_1,x_2,x_3)$ is a homog. 2-cocycle for $SL_3(\mathbb{Z})$.

- **Q** : What is this 2-cocycle ? Is it trivial ?

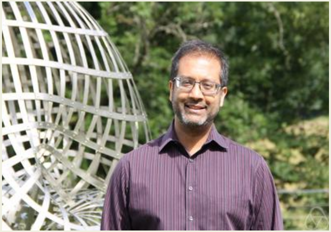
Theorem

(P.C.) This 2-cocycle on $SL_3(\mathbb{Z})$ is equal to the "Shintani-cocycle" recently introduced in C-Dasgupta-Greenberg following work of Solomon, as a variation of the Sczech-cocycle (1993). It is non-trivial because of the connection to values of zeta functions.

Proof.

The cocycles all match up using explicit formulas. The non-triviality also follows from [CDG2015] : if $F/\mathbb{Q}$ is cubic and tot. real, then 2 fund. units of F act on $\mathcal{O}_F$, and give rise to a 2-cycle in homology. Conclude invoking Shintani : $< P^k, C_F >= \zeta_F(1-k) \in \mathbb{Q}^\times$. □

Eisenstein cocycle on $SL_n(\mathbb{Z})$ Hall of Fame.



Samit Dasgupta Matthew Greenberg

Robert Sczech

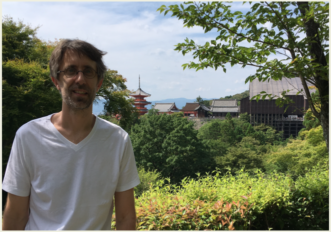
Complex cubic fields...?

Let $F/\mathbb{Q}$ be a complex cubic field. The unit group $U_F = \pm < \epsilon_F >$ has rank 1 and acts on $\mathcal{O}_F = \omega_1\mathbb{Z} + \omega_2\mathbb{Z} + \omega_3\mathbb{Z}$, and as such gives rise to a 1-cycle in the homology of $SL_3(\mathbb{Z})$.

It is very tempting to make a conjecture about the algebraicity of the F_{ab} evaluated on this 1-cycle. By lack of numerical evidence at this early stage, I'll be cautious and will not make any precise claim yet. Avoiding any ridicule.

Thanks, references, and advertisement for related work-in-progress:

• **"Euler classes transgressions and Eisenstein cohom. of $GL_n(\mathbb{Z})$."**
joint with Nicolas Bergeron, Luis Garcia and Akshay Venkatesh.



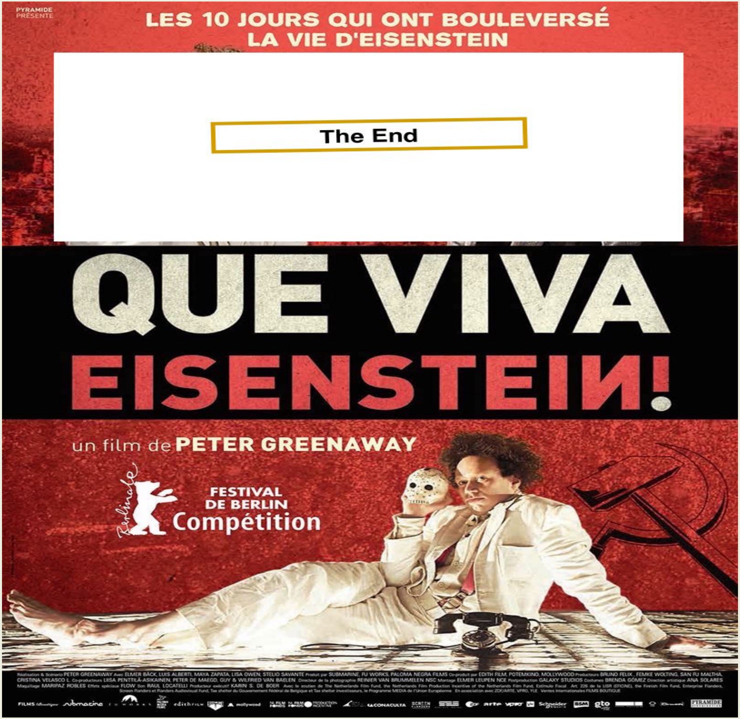
• **Takagi Lectures by N. Bergeron (incl. 55p. notes + video).**

• **see also his IAS video March 2018.**

• **technicalities involved : similar to yesterday's talk by S. Sankaran**

• **talk this afternoon by H. Darmon.**

The End


PYRAMIDE PRÉSENTE
LES 10 JOURS QUI ONT BOULEVERSÉ LA VIE D'EISENSTEIN
QUE VIVA EISENSTEIN!
un film de PETER GREENAWAY
Berlinale
FESTIVAL DE BERLIN Compétition