On the density of zeros of the Riemann zeta-function near the critical line

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CNTA XV
10 July 2018

The Riemann zeta-function

- Defined by $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $\text{Re}(s) > 1$

- Defined over $\mathbb{C}$ by analytic continuation and the functional equation

$$\zeta(s) = \chi(s)\zeta(1-s), \quad \text{where} \quad \chi(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi}{2}s\right)\Gamma(1-s)$$

Conjecture (Riemann, 1859)

All the non-real zeros of $\zeta(s)$ have real part $1/2$.

- Partial progress: not many zeros have real part $\neq 1/2$. This can lead to results that are almost as good as those implied by RH.

Zero-density estimates

Let $N(\sigma, T) =$ the number of zeros with real part $> \sigma$ and imaginary part between 0 and T.

Theorem (Ingham, 1937): If the bound $\zeta(\frac{1}{2} + it) \ll t^b$ holds, then

$$N(\sigma, T) \ll T^{2(1+2b)(1-\sigma)+\varepsilon} \text{ uniformly for } \frac{1}{2} \leq \sigma \leq 1,$$

which in turn implies that

$$p_{n+1} - p_n \ll p_n^{\vartheta} \qquad \text{with} \qquad \vartheta = \frac{1+4b}{2+4b} + \varepsilon$$

The density hypothesis

Conjecture (folklore)

$$N(\sigma, T) \ll T^{A(1-\sigma)+\varepsilon} \quad \textit{with} \quad A \le 2, \quad \textit{for} \quad \frac{1}{2} \le \sigma \le 1$$

- Ingham (1940): $A \le \frac{3}{2-\sigma}$ for $\frac{1}{2} \le \sigma \le 1$
- Montgomery (1971): $A \le \frac{2}{\sigma}$ for $\frac{4}{5} \le \sigma \le 1$
- Huxley (1972): $A \le \frac{3}{3\sigma - 1}$ for $\frac{3}{4} \le \sigma \le 1$
- Jutila (1977): $A \le 2$ for $\frac{11}{14} \le \sigma \le 1$
- Bourgain (2000): $A \le 2$ for $\frac{25}{32} \le \sigma \le 1$

Best estimates for σ close to $1/2$

Claim

$$N(\sigma, T) \ll T^{1+\theta(1-2\sigma)} \log T \quad \textit{uniformly for} \quad \frac{1}{2} \leq \sigma \leq 1.$$

- Selberg (1946): claim is true for all $0 < \theta \leq \frac{1}{8}$
- Jutila (1983): claim is true for all $0 < \theta < \frac{1}{2}$
- density hypothesis $\Leftrightarrow$ claim is true for all $0 < \theta < 1$
- Conrey (1989): claim is true for all $0 < \theta < \frac{4}{7}$

...but he didn't publish a proof

The standard approach: estimate $\displaystyle\int_T^{2T} |\zeta(\sigma + it)M(\sigma + it)|^2 \, dt$

Conrey's theorem

Theorem (Conrey, 1989; B, 2017)

If $0 < \theta < \frac{4}{7}$, *then*

$$N(\sigma, T) \ll T^{1+\theta(1-2\sigma)} \log T$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1$.

This follows from

Theorem (B, 2017)

For a suitable mollifier $M(\sigma + it)$ *and fixed* $0 < \theta < \frac{4}{7}$, *we have*

$$\frac{1}{T}\int_T^{2T} |\zeta(\sigma + it)M(\sigma + it)|^2 \, dt = 1 + O\left(T^{\theta(1-2\sigma)}\right)$$

uniformly for $\frac{1}{2} + \frac{1}{\theta \log T} \leq \sigma \leq \frac{1}{2} + \varepsilon$.

Brief sketch of Conrey's technique

Let $M(s) = \sum_k \frac{a_k}{k^s}$. Then

$$\frac{1}{T\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(t-T)^2}{T^2}\right) |\zeta(\sigma+it)M(\sigma+it)|^2 \, dt$$

$$\approx \sum_{h,k} \frac{a_h a_k}{k^{2\sigma}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{1-2\sigma}} \times$$

$$\times \frac{1}{iT\sqrt{\pi}} \int_{2-i\infty}^{2+i\infty} \exp\left(\frac{(s-\sigma-iT)^2}{T^2}\right) \chi(2\sigma-s) \left(\frac{mnh}{k}\right)^{-s} \, ds$$

$$\approx \sum_{h,k} \frac{a_h a_k}{k^{2\sigma}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{1-2\sigma}} \times$$

$$\times \left(\frac{mnh}{k}\right)^{1-2\sigma} \int_0^{\infty} z^{-\sigma+iT} \exp\left(-\frac{T^2}{4}(\log z)^2\right) \exp\left(2\pi i \frac{mnh}{k} z\right) \, dz$$

$$+ \cdots$$

Estermann's functional equation (1930)

For relatively prime integers H and K, define

$$D(s,\alpha,\beta,H/K) \;=\; \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^{s+\alpha}n^{s+\beta}}\exp\left(2\pi i\frac{mnH}{K}\right).$$

Then we have the functional equation

$$\begin{aligned} D\left(s,\alpha,\beta,H/K\right) \;=\; & \frac{1}{\pi}\left(\frac{K}{2\pi}\right)^{1-2s-\alpha-\beta}\Gamma(1-s-\alpha)\Gamma(1-s-\beta)\times \\ & \times\Big\{D\left(1-s,-\alpha,-\beta,\overline{H}/K\right)\cos\left(\frac{\pi}{2}(\alpha-\beta)\right) \\ & \quad - D\left(1-s,-\alpha,-\beta,-\overline{H}/K\right)\cos\left(\frac{\pi}{2}(2s+\alpha+\beta)\right)\Big\}, \end{aligned}$$

where $\overline{H}$ is defined by $\overline{H}H \equiv 1 \pmod{K}$.

Main tools: Kloosterman sum estimates

- Weil (1948): If I is an interval of length at most c, then

$$\sum_{\substack{a\in I\\(a,c)=1}} \exp\left(2\pi i \frac{m\overline{a}}{c}\right) \ll (m,c)^{1/2}c^{1/2+\varepsilon}.$$

- Deshouillers and Iwaniec (1984): If $C(a,\ell) \ll 1$ then

$$\sum_{B<b\le 4B}\ \sum_{\substack{V<v\le 2V\\(b,v)=1}} \left|\sum_{\substack{A<a\le 2A\\(a,v)=1}}\ \sum_{N<\ell\le 2N} C(a,\ell)\exp\left(2\pi i\frac{\ell\overline{ab}}{v}\right)\right| \ll$$

$$(ABNV)^{\frac{1}{2}+\varepsilon}\left\{(VB)^{\frac{1}{2}}+(N+A)^{\frac{1}{4}}\left[VB(N+A)(V+A^2)+NA^2B^2\right]^{\frac{1}{4}}\right\}.$$

A non-standard mollifier (B, 2017)

$$M(s) = \sum_{k \le T^{\theta+\varepsilon}} \frac{\mu(k)}{k^s} \left(\frac{W(T^\theta/k)}{W(T^\theta)} \right),$$

where

$$W(\xi) \;=\; \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(s+1) \frac{\zeta(2\sigma+s)}{\zeta(2\sigma)} \zeta(1+s) \xi^s \, ds.$$

- allows the evaluation of the main terms with relative ease
- lets us apply Conrey's technique to bound the error terms
- a similar $W(\xi)$ was used by Soundararajan to study L-functions