Shift Ramanujan expansions

(jointly with Ram Murty)

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We define the *correlation* (*"shifted convolution sum"*) of any couple $f, g : \mathbf{N} \to \mathbf{C}$ as

$$(1) \qquad C_{f,g}(N,a) \stackrel{def}{=} \sum_{n \le N} f(n) g(n+a).$$

The integer $a > 0$ is the *shift*. Classic heuristic:

$$(2) \qquad C_{f,g}(N,a) \sim S_{f,g}(a) N,$$

defining the f and g *singular series* ($\widehat{f}, \widehat{g}$, soon):

$$S_{f,g}(a) \stackrel{def}{=} \sum_{q=1}^{\infty} \widehat{f}(q) \widehat{g}(q) c_q(a),$$

where the *Ramanujan sum* $c_q(a)$ is defined, with $e_q(m) \stackrel{def}{=} e^{2\pi i m/q}$ the *additive characters*:

$$(3) \quad c_q(a) \stackrel{def}{=} \sum_{\substack{j \le q \\ (j,q)=1}} \cos\left(\frac{2\pi j a}{q}\right) = \sum_{j \in \mathbf{Z}_q^*} e_q(ja).$$

From Möbius inversion, $f' \stackrel{def}{=} f * \mu$ & $g' \stackrel{def}{=} g * \mu$

$$\Rightarrow \quad f(n) = \sum_{d|n} f'(d) \ \ \& \ \ g(m) = \sum_{q|m} g'(q),$$

so : **vital remark** is that inside their correlation our $f(n), g(m)$ become *truncated divisor sums*

$$\sum_{d|n, d \le N} f'(d), \qquad \sum_{q|m, q \le N+a} g'(q)$$

depending on both variables, N and shift a. The condition "$d|n$" may be expressed as

$$\text{(4)} \qquad \mathbf{1}_{d|n} = \frac{1}{d} \sum_{j \le d} e_d(jn) = \frac{1}{d} \sum_{q|d} c_q(n),$$

from the *orthogonality of additive characters* and g.c.d. rearrangement. Hence, inside $C_{f,g}$ we have *finite Ramanujan expansions*, say

$$f(n) = \sum_{q \le N} \widehat{f}(q) c_q(n), \ g(m) = \sum_{q \le N+a} \widehat{g}(q) c_q(m)$$

with, say, *finite Ramanujan coefficients*

$$\widehat{f}(q) \stackrel{def}{=} \sum_{\substack{d\le N\\ d\equiv 0 \bmod q}} \frac{f'(d)}{d}, \quad \widehat{g}(q) \stackrel{def}{=} \sum_{\substack{d\le N+a\\ d\equiv 0 \bmod q}} \frac{g'(d)}{d}.$$

See, now the singular series is a **singular sum**

$$S_{f,g}(a) = \sum_{q\le N} \widehat{f}(q)\widehat{g}(q)c_q(a).$$

Dependence on N here is actually inside f, g. The $a-$dependence in $\widehat{g}(q)$ is overridden for g

$$g(m) = g_Q(m) \stackrel{def}{=} \sum_{d|m, d\le Q} g'(d),$$

a **truncated divisor sum**(t.d.s.) *of range* Q, where $Q \le N$ is $a-$independent. We'll also assume : $a-$dependence is not in f, g, nor in their supports, say, $C_{f,g}(N, a)$ is **fair**.

These two are our **basic hypotheses**, we'll assume implicitly henceforth.

Our most important finding with Ram Murty is the **R**amanujan **e**xact **e**xplicit **f**ormula: **(Reef)**

$$C_{f,g}(N,a) = \sum_{q \le N} \frac{\widehat{g}(q)}{\varphi(q)} \sum_{n \le N} f(n) c_q(n) c_q(a),$$

with $\varphi(q) \stackrel{def}{=} |\{j \le q : (j,q) = 1\}|$ the well known *Euler function*.

The **Reef** holds iff some equivalent conditions hold: it's **not for free**. Actually, we gave on JNT the following characterizations. Call

$$C_{f,g}(N,a) = \sum_{\ell=1}^{\infty} \widehat{C_{f,g}}(N,\ell) c_\ell(a), \quad \forall a \in \mathbf{N}$$

the **shift-Ramanujan expansion** (s.R.e.). We wish to use the *Carmichael formula*

$$\widehat{C_{f,g}}(N,q) = \frac{1}{\varphi(q)} \lim_{x \to \infty} \frac{1}{x} \sum_{a \le x} C_{f,g}(N,a) c_q(a);$$

also we call s.R.e. **pure** iff the $\widehat{C_{f,g}}(N,\ell)$ don't depend on a, as we expect. We will call s.R.e. **uniform** when it converges uniformly in $a \in \mathbf{N}$.

Roughly speaking, the following is quoting our Theorem 1 (Coppola-Murty on JNT).

(F.A.E.=Following Are Equivalent)

In the basic hypotheses (see above), F.A.E.:

(i) s.R.e. is uniform & pure

(ii) Carmichael formula holds

(iii) the Reef holds

(iv) the correlation is a t.d.s., resp.to a

(See, (iv) on JNT, "s.R.e. is finite and pure", is equivalent to: "$C_{f,g}(N,a)$ t.d.s.", here)

As **a kind of "gift"** (under suitable, additional & very natural hypotheses) The Reef implies classic heuristic (2) for the correlation: this is our, say, Coppola-Murty Corollary 1 on JNT.

The *Eratosthenes Transform* of our $C_{f,g}$ is

$$C'_{f,g}(N,d) \stackrel{def}{=} \sum_{t|d} C_{f,g}(N,t)\mu\left(\frac{d}{t}\right), \quad \forall d \in \mathbf{N}$$

and as usual $\omega(d) \stackrel{def}{=} |\{p \text{ prime} : p \text{ divides } d\}|$.

We may prove, in full generality, compare `arxiv` for $2k-$twin primes case, the following further characterization for the Reef.

In our basic hypotheses, Reef's equivalent to Delange Hypothesis (for our correlation), say

$$(DH) \qquad\qquad \sum_{d} \frac{2^{\omega(d)}}{d} \left| C'_{f,g}(N,d) \right| < \infty.$$

In fact, in the same spirit of our Corollary 1 (proving heuristics above), we got a *Conditional Proof of Hardy Littlewood Conjecture* (for $a = 2k$ twin primes), under Delange Hypothesis (DH) for twin primes (see my `arxiv`).

THANKS!!!

[1] Coppola, G., Murty, M.Ram and Saha, B. - *Finite Ramanujan expansions and shifted convolution sums of arithmetical functions* - JNT(2017)

[2] Coppola, G. and Murty, M.Ram - *Finite Ramanujan expansions and shifted convolution sums of arithmetical functions, II* - JNT(2018)

[3] Coppola, G. - *An elementary property of correlations* - `arxiv:1709.06445`