Rational Equivalences on Surfaces using Hyperelliptic Subcurves

Jonathan Love*
joint with Evangelia Gazaki**

*McGill University, CRM-ISM
**University of Virginia, NSF

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Zero-Cycles

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Definition

A *zero-cycle* on X is a formal sum of the form

$$a_1[P_1] + \cdots + a_n[P_n],$$

where $a_i \in \mathbb{Z}$ and P_i are closed points of X.

Rational Equivalence

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Definition

A zero-cycle is a rational equivalence (or rationally equivalent to 0) if it can be written as a linear combination of divisors of rational functions on curves in X.

$$\mathrm{CH}_0(X) := (\text{Group of zero-cycles on } X)/(\text{rational equivalence})$$
$$A_0(X) := (\text{zero-cycles with } \textstyle\sum a_i \deg(P_i) = 0)/(\text{rational equivalence}).$$

Main Problem

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Given a zero-cycle z on X, determine whether it is a rational equivalence.

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There is a surjection $\Sigma : A_0(X) \to \operatorname{Alb} X(k)$:

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Wishful Thinking

The map Σ is an isomorphism.

If true, a zero-cycle is a rational equivalence if and only if it sums to 0 in $\mathrm{Alb}\, X(k)$.

Positive and Negative Results

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Theorem 1[1]

If $k = \overline{\mathbb{F}_p}$ then $\ker \Sigma = 0$ (Σ is an isomorphism).

[1] Spencer Bloch. "An example in the theory of algebraic cycles". In: *Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976)*. 1976, 1–29. Lecture Notes in Math., Vol. 551, Attributed to Richard Swan.

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Theorem 2[2]

If $k = \mathbb{C}$, then $\ker \Sigma$ is infinite-dimensional.

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[2] D. Mumford. "Rational equivalence of 0-cycles on surfaces". In: *J. Math. Kyoto Univ.* 9.2 (1969), pp. 195–204.

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Theorem 3[3]

If $k = \mathbb{Q}_p$ and E_1, E_2 have ordinary good reduction, then $\ker \Sigma$ is a finite group times a divisible group.

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[2] Mumford, "Rational equivalence of 0-cycles on surfaces".

[3] Evangelia Gazaki and Isabel Leal. "Zero Cycles on a Product of Elliptic Curves Over a p-adic Field". In: *International Mathematics Research Notices* (Mar. 2021). ISSN: 1073-7928.

Let X be an abelian variety over a field k.

[4] Spencer Bloch. "Algebraic cycles and values of L-functions.". In: *Journal für die reine und angewandte Mathematik* 350 (1984), pp. 94–108.

[5] A.A. Beilinson. "Higher regulators and values of L-functions". In: *J Math Sci* 30 (1985), pp. 2036–2070.

Bloch-Beilinson Conjecture

Let X be an abelian variety over a field k.

Conjecture 4 (BB)[4,5]

If k is a number field, then $\ker \Sigma$ is a finite group.

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One problem: X has points of arbitrarily large degree. Idea: restrict to zero-cycles generated by a smaller set of points.

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[5] Beilinson, "Higher regulators and values of L-functions".

A Weak Variant of Bloch-Beilinson

Let $C_1, \ldots, C_d$ be curves over k, and $X = C_1 \times \cdots \times C_d$. Let $\pi_i : X \to C_i$ be the projection map.

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Definition

Let z_i be a closed point of C_i The subgroup of $\mathrm{CH}_0(X)$ generated by zero-cycles of the form $\pi_1^*(z_1) \cap \cdots \cap \pi_d^*(z_d)$ is the componentwise subgroup $A_{comp}(X)$.

Note that $A_{comp}(X)$ contains all zero-cycles supported on $X(k)$.

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$$((\mathbb{Z} \times J_1(k)) \otimes \cdots \otimes (\mathbb{Z} \times J_d(k))/\mathbb{Z} \twoheadrightarrow A_{comp}(X) \twoheadrightarrow J_1(k) \times \cdots \times J_d(k)$$

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Conjecture 5 (WBB)

If k is a number field, then $\ker \Sigma \cap A_{comp}(X)$ is finite.

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3. Prasanna and Srinivas: E_1 and E_2 both have conductor 37, or both have conductor 91. Uses Heegner points on a common modular parametrization.[6]

[6] Kartik Prasanna and Vasudevan Srinivas. "Zero Cycles on a Product of Elliptic Curves". Private correspondence. 2018.

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4. E_1 is a rank 1 curve of the form $y^2 = x^3 - 3t^2x + b$, with no torsion point with x-coordinate equal to t; E_2 is any rank 1 curve in certain a one-parameter family depending on E_1. Uses rational curves in the Kummer surfaces of $E_1 \times E_2$.[7]

[6] Prasanna and Srinivas, "Zero Cycles on a Product of Elliptic Curves".

[7] Jonathan Love. "Rational Equivalences on Products of Elliptic Curves in a Family". In: *Journal de Théorie des Nombres de Bordeaux* 32.3 (2020), pp. 923–938. ISSN: 12467405, 21188572.

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What about products of three or more curves? Elliptic curves of higher rank? Higher genus curves?

[6] Prasanna and Srinivas, "Zero Cycles on a Product of Elliptic Curves".

[7] Love, "Rational Equivalences on Products of Elliptic Curves in a Family".

More curves in the product

Theorem 6 (Gazaki-L. '22)

Let S be a set of smooth projective curves over k such that $C(k) \neq \emptyset$ for all $C \in S$, and all pairs $C, C' \in S$ satisfy BB (resp. WBB). If $X = C_1 \times \cdots \times C_d$ with $C_i \in S$ for all $i \in \{1, \ldots, d\}$, then X satisfies BB (resp. WBB).

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Proof Idea: Raskind and Spiess[8] established an isomorphism

$$A_0(X) \simeq \prod_{\nu=1}^{d} \prod_{1 \leq i_1 < \cdots < i_\nu \leq d} K(k; J_{i_1}, \ldots, J_{i_\nu}),$$

where $K(k; J_{i_1}, \ldots, J_{i_\nu})$ are Somekawa K-groups. Use the defining relations of these K-groups to prove a product formula.

[8] Wayne Raskind and Michael Spiess. "Milnor K-groups and zero-cycles on products of curves over p-adic fields". In: *Compositio Math.* 121.1 (2000), pp. 1–33. ISSN: 0010-437X.

A useful lemma

Let C_1, C_2 be smooth curves over k, with fixed k-rational points O_1, O_2. Let J_i denote the Jacobian variety of C_i. The Albanese variety of $C_1 \times C_2$ is $X := J_1 \times J_2$.

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$$\begin{aligned} D_{P,Q} &:= ([P] - [O_1]) \otimes ([Q] - [O_2]) \\ &= [(P, Q)] - [(P, O_2)] - [(O_1, Q)] + [(O_1, O_2)]. \end{aligned}$$

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Proof Idea: Use a "diagonal" map $H \to X$ given by $P \mapsto (\phi_1(P), \phi_2(P))$, as well as "vertical/horizontal" maps $P \mapsto (\phi_1(P), Q)$ and $P \mapsto (Q, \phi_2(P))$.

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Since $[P] + [\iota(P)] - 2[W]$ is principal, its pushforward along any of these maps is a rational equivalence; find a combination that equals a multiple of $D_{\phi_1(P),\phi_2(P)}$.

Applications

This lemma subsumes the previous results:

- Prasanna-Srinivas: For $N = 37$ or $N = 91$, $X_0(N)$ is hyperelliptic. Use modular parametrizations $\phi_i : X_0(N) \to E_i$ and a Heegner point $P \in X_0(N)(\mathbb{Q})$.

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- L.: A rational curve in the Kummer surface lifts to a hyperelliptic curve in $E_1 \times E_2$.

An example with k an imaginary quadratic field K:

Corollary

Let $E/\mathbb{Q}$ such that $E_{\overline{\mathbb{Q}}}$ has CM by $\mathcal{O}_K$. If $E(\mathbb{Q})$ has rank 1 and $E(K)$ has rank 2, then $(E \times E)_K$ satisfies WBB.

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Examples with higher genus curves:

Corollary

Let H be a hyperelliptic curve over k with Jacobian J. Suppose $J(k)$ has rank 1, and there exist $P, W \in H(k)$ with W a Weierstrass point and $[P] - [W]$ of infinite order. Then $H \times H$ satisfies WBB.

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LMFDB has 860 genus 2 curves over $\mathbb{Q}$ with conductor ≤ 10000 satisfying the above conditions.

Applications

Given $E_1, E_2/\mathbb{Q}$, we can construct a hyperelliptic curve H with Jacobian isogenous to $E_1 \times E_2$. Then each $P \in H(\mathbb{Q})$ gives a rational equivalence on $E_1 \times E_2$.

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If we can find $(\operatorname{rk} E_1(\mathbb{Q})) \cdot (\operatorname{rk} E_2(\mathbb{Q}))$ independent rational equivalences, then $E_1 \times E_2$ satisfies WBB.

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# of E_1	rk $E_1(\mathbb{Q})$	# of E_2	rk $E_2(\mathbb{Q})$	total # of pairs	# WBB
100	1	100	1	4950	2602
100	1	100	2	10000	3311
500	1	20	3	10000	955
100	2	100	2	4950	995
500	2	20	3	10000	615
20	3	20	3	190	17

Table: Number of pairs of elliptic curves over $\mathbb{Q}$ provably satisfying WBB.

Further work

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Some pairs of curves (e.g. Cremona labels 37a1 and 43a1) have been shown to satisfy WBB using bielliptic curves (a degree 2 cover of an elliptic curve). So in general, more complicated curves and/or principal divisors on them may be necessary.

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Question

Are hyperelliptic curves sufficient to generate every rational equivalence in $A_{comp}(X)$, at least when $X = E_1 \times E_2$?

Related to a conjecture of Bogolomov: every $\overline{k}$-rational point on a K3 surface X/k lies on a rational curve in X defined over $\overline{k}$.[9]

[9] Fedor Bogomolov and Yuri Tschinkel. "Rational Curves and Points on K3 Surfaces". In: *American Journal of Mathematics* 127.4 (2005), pp. 825–835. ISSN: 00029327, 10806377.

Thank you for listening! Any questions?