Higher Kato elements and equivariant complexes

Peter Xu

Department of Mathematics and Statistics
McGill University

October 15, 2022

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  - Example: diagonal restrictions of Hilbert modular surface $\mathcal{Y} \subset \mathcal{Y}_F$.

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- Relation to Eisenstein series:

$$ d \log {}_{\mathcal{C}}g = E_1^{(\mathcal{C})}(\tau, z)dz + E_2^{(\mathcal{C})}(\tau, z)d\tau $$

Kato elements and Eisenstein series

- Cup products in motivic cohomology:

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  - Motivic classes bound Selmer groups

The general situation

- If we have a genus-g abelian family $\mathcal{A} \to \mathcal{S}$, can construct "big Eisenstein class" by specifying torsion section residues in any suitable cohomology theory:

$$\dots \to H^{2g-1}(\mathcal{A}) \to H^{2g-1}(\mathcal{A} - C) \to H^0(C) \to H^{2g}(\mathcal{A}) \to \dots$$

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- Can construct equivariant theory by taking equivariant versions of the complexes; e.g. Bloch's cycle complex for motivic cohomology, Dolbeault complex for coherent cohomology. Get cocycles $\Theta^M_{\mathcal{C}}$, $\Theta_{\mathcal{C}}$

Application: Diagonal restrictions of Hilbert-Eisenstein series

- In the case of $\mathcal{E}^2 \to \mathcal{Y}$ (Sharifi-Venkatesh) equivariant complex is computable:

$$0 \to K_2^M(\mathbb{Q}(\mathcal{E}^2))^{(0)} \to \left(\oplus_{D\in\mathcal{Y}^{(1)}} K_1^M(\mathbb{Q}(D))\right)^{(0)} \to \left(\oplus_{x\in\mathcal{Y}^{(2)}} K_0^M(\mathbb{Q}(x))\right)^{(0)}$$

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- Upshot: compute some interesting values of $\Theta_{\mathcal{C}}$: e.g. when $\gamma = \gamma_F$ primitive hyperbolic for a real quadratic field F, get diagonal restriction of weight $(1,1)$-Hilbert-Eisenstein series

$$\Theta_{\mathcal{C},x}(\gamma_F) \sim \sum_{(m,n)\in\mathcal{O}_F^2/\Delta} \frac{1}{(m\tau+n+x)(m'\tau+n'+x')}$$

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realizing each Stark-Heegner point ("almost algebraicity").

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- Kolyvagin-type applications

Conclusion, future directions

- Kudla program philosophy: derivatives at central values of incoherent Eisenstein series are related to algebraic cycles
  - Relation to constructions on p-adic symmetric spaces, the p-adic Kudla program
  - Applications to special values of L-functions
- Euler systems for other Shimura varieties