An Excised Orthogonal Model for Families of Cusp Forms

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Joint work with Owen Barrett, Astrid Lilly, and Steven J. Miller

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L-Functions Connection to Random Matrix Theory

Katz–Sarnak Philosophy

In the limit, statistics of L-functions match statistics for large random matrices from particular classical compact groups.

- $U(N)$
- $O(N)$
- $USp(2N)$
- $SO(2N)$

Modeling Low Lying Zeros of L-functions

Theorem (Kohnen-Zagier)

- $f \in S_k$
- $g \in S^+_{(k+1)/2}$ *Shimura correspondence*
- $(-1)^k d > 0$
- ψ_d *Kronecker character*

Then

$$L_f(\tfrac{1}{2}, \psi_d) = \kappa_f \frac{c(|d|)^2}{|d|^{(k-1)/2}}, \text{ where } \kappa_f = \frac{(k-1)!}{\pi^{k/2}} \frac{\langle f, f \rangle}{\langle g, g \rangle}$$

where $c(|d|)$ *is the* $|d|^{th}$ *Fourier coefficient of* g.

Excised Ensemble and Cut-Off Value

Definition

An *excised ensemble* is a collection of random matrices in which we remove any generated matrix whose characteristic polynomial evaluated at 1 is less than a cut-off value.

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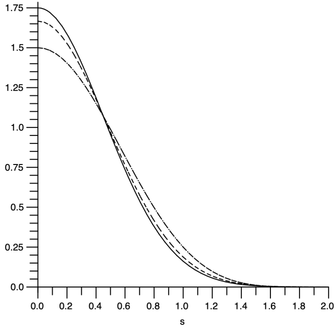
We use excised ensembles of random matrices to model L-function statistics for *finite* conductor.

Non-excised RMT Ensemble

Without excision, the orthogonal model does not capture any of the repulsion behavior.



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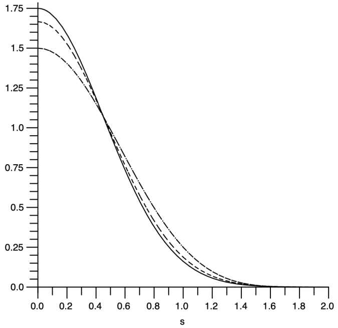
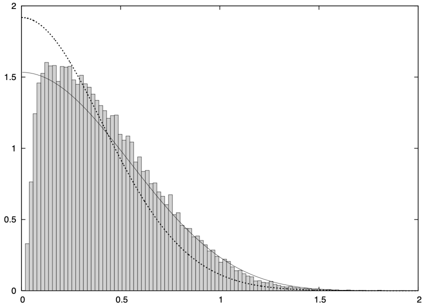




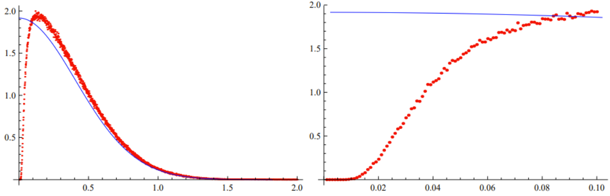
Left: Probability density of normalized eigenvalue closest to 1 for SO(8) (solid), SO(6) (dashed) and SO(4) (dot-dashed).
Right: Distribution of lowest zero for $L_{E_{11}}(s, \chi_d)$ with $0 < d \leq 400,000$ compared to two sizes of non-excised random matrix ensembles.

The Excised Orthogonal Ensemble

In 2011, Dueñez, Huynh, Keating, Miller, and Snaith developed the *excised orthogonal ensemble*, a sub-ensemble of $SO(2N)$, as a random matrix analogue for the family of quadratic twists of a given elliptic curve.

Repulsion of First Eigenvalue

The excised orthogonal ensemble exhibits the desired repulsion in the distribution of first eigenvalues:



Left Image: Distribution of first eigenvalues from non-excised SO(24) random matrices (blue) versus excised SO(24) random matrices (red). For the excised plot, the sample size before excision was 3,000,000, cutoff approximately 0.005.

Right Image: Enlargement of data near the origin.

Properties of the Excised Ensemble

On the scale of mean spacing, the excised ensemble exhibits an exponentially small hard gap determined by the cut-off value, with soft repulsion on a larger scale.

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Taking $N \to \infty$, there is limiting orthogonal behavior, which qualitatively agrees with Miller's discrepancy.

Computing the Cutoff Value

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Dueñez, Huynh, Keating, Miller, Snaith determined this cut-off value numerically by using the proportion of quadratic twists with central vanishing.

A Weight 4 Newform

Let f be the normalized, weight 4, level 7 newform over $\mathbb{Q}$ (Label 7.4.a.a in LMFDB).

The Fourier coefficients $c(d)$ of the corresponding weight $5/2$ form can be obtained by adding the values of a quadratic form over a 3-dimensional lattice.

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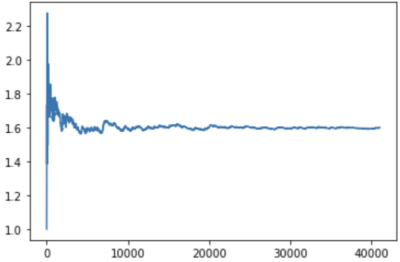

Figure: The running average for $c(d)^2/d^{3/2}$ for positive fundamental discriminants such that $c(d) \neq 0$. This value is proportional to the central value by Kohnen–Zagier.

Lowest-lying Zeros

Using PARI/GP software, we computed the first three zeros for many twists of f by quadratic characters.

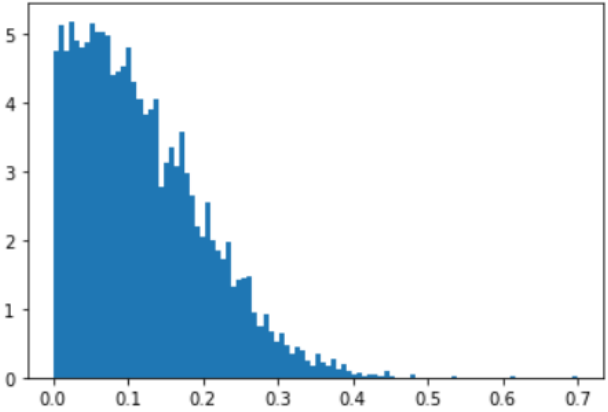

Figure: The distribution of lowest-lying zeros for all twists of f by positive fundamental discriminants $d \leq 41128$ with $\gcd(7, d) = 1$ such that the twisted L-function does not vanish at the central point.

Effective Matrix Size

To model the finite-conductor statistics, we need to pick a finite matrix size for our $SO(2N)$ ensemble.

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We choose a matrix size to equate the mean densities of eigenvalues with the mean density of L-function zeros, giving

$$N_{std} \;=\; \log\left(\frac{\sqrt{M}X}{2\pi e}\right) \;\approx\; 8,$$

where $M = 7$ is the level of f and $X = 41128$ is the largest twist we consider in our statistics.

The Orthogonal Ensemble

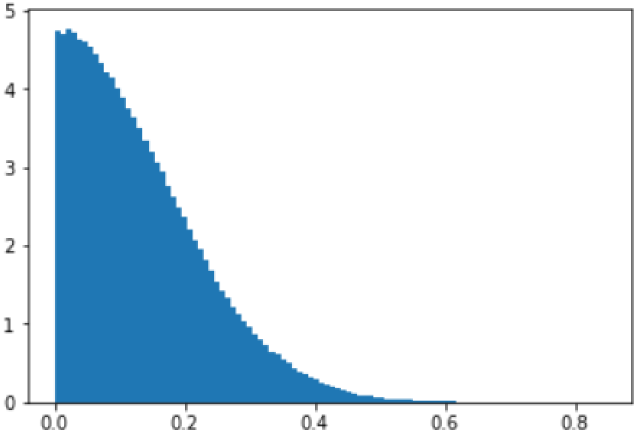

Figure: The distribution of the first eigenvalue for 10^6 random matrices in $SO(16)$.

Cut-off Computation

If we excise our random matrix data by a cutoff value C, we obtain a new distribution.

To find an optimal cut-off value, we compute the L^1 distance between the cumulative distribution functions of the excised matrix distribution and the distribution of zeros.

Cut-off Computation

When we use all of our lowest-lying zero data for $d \leq 41128$, we obtain an optimal cut-off value of

$$C = 0.00095\ldots$$

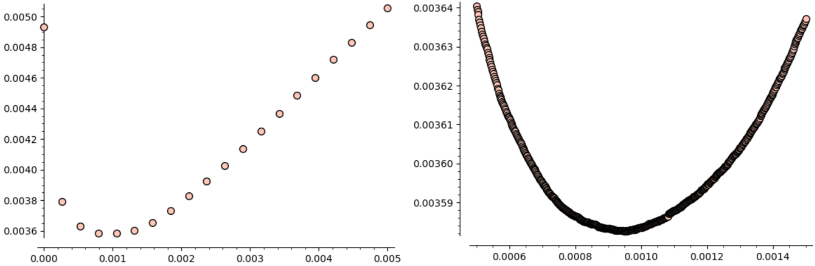

Figure: (Left) The L^1 distance between our (rescaled) excised orthogonal data and our lowest-lying zero data for various cutoffs c. (Right) The same data zoomed into the global minimum.

Mock Probability Distribution

Let $P_{O^+}(N,x)$ be the probability density of $x = \Lambda_A(1,N)$ over the ensemble $A \in SO(2N)$. We construct a mock probability density $P_f(d,x)$ of $x = L_f(\frac{1}{2},\psi_d)$ at the d^{th} Kronecker twist via

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$$P_{O^+}(N,x) = \frac{1}{2\pi i x}\int_{(c)} M_{O^+}(N,s)x^{-s}\,ds$$

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The main feature is that, up to first order,

$$P_f(d,x) \;\sim\; a_f(-1/2)P_{O^+}(\log d,x).$$

Mock Probability Distribution

To compute the cutoff value arithmetically, we need a scaling factor to translate from the RMT perspective to the L-function perspective.

$$\begin{aligned} a_f(s) \ &= \ \left[\prod_p \left(1-\frac{1}{p}\right)^{s(s-1)/2}\right] \\ &\times \left[\prod_{p\nmid M}\left(1+\frac{1}{p}\right)^{-1}\left(\frac{1}{p}+\frac{1}{2}\left[\mathcal{L}_p(p^{-1/2},f)^s+\mathcal{L}_p(-p^{-1/2},f)^s\right]\right)\right] \\ &\times \ \mathcal{L}_M\left(\frac{\varepsilon_f}{M^{1/2}}\right)^s \end{aligned}$$

Constructing a cut-off

On the other hand, by the discretization of central value, the value is forced to be 0 if it's less than some constant.

Lemma (Discretization)

$$L_f(k, \psi_d) < \frac{\kappa_f}{|d|^{(k-1)/2}} \implies L_f(k, \psi_d) = 0$$

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Lemma (Discretization)

$$L_f(k,\psi_d) < \frac{\kappa_f}{|d|^{(k-1)/2}} \implies L_f(k,\psi_d) = 0$$

Note that this lemma is merely using the fact that $c(d)$s are integers, and the $\frac{\kappa_f}{|d|^{(k-1)/2}}$ bound is the crudest possible(taking 1). Ideally, we can do better than 1.

Conjecture

$$|\{L_f(s,\psi_d) \in \mathcal{F}_f^+(X) : d \text{ prime}, L_f(k,\psi_d) = 0\}|$$
$$\sim \sum_{\substack{d\le X \\ d \text{ prime}}} Prob(0 \le Y_d \le \tfrac{\kappa_f \delta_f}{d^{(k-1)/2}})$$

where δ_f is some constant related to the discretization.

Cutoff value for RMT model

We first obtain our standard matrix size N_{std} for our RMT model:

$$N_{std} = \log\left(\frac{\sqrt{M}X}{2\pi e}\right) \sim log(X).$$

And by connecting the probability distribution function of central value of this family of L-function and the corresponding characteristic polynomial of RMT model evaluate at 1, we obtain the following relation

$$P_f(d,x) \ \sim \ a_f(-1/2)P_{O^+}(\log d, x).$$

Cutoff value for RMT model

To obtain the cutoff for RMT model: C_{std}, we normalize our cutoff $\frac{\kappa_f \delta_f}{d^{(k-1)/2}}$ for the pdf of L-function by applying the above relation of the pdfs :

Theorem

$$C_{std} = a_f^{-2}(-1/2)\delta_f \kappa_f \times exp((1-k)N_{std}/2)$$

Conjecturing a cutoff

To get good δ_f, we conjecture this quantity should grow accordingly with the scale of $c(d)$s. Thus we first chose $\delta_f = \mathbb{E}(c(d)^2)$ regarding the Kohnen–Zagier formula.

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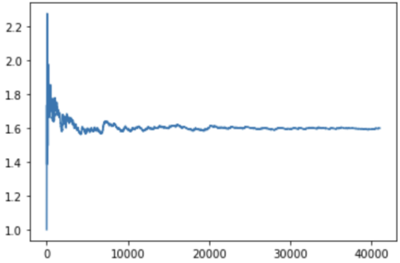

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Ongoing Work

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Compute with more weight 4 forms to see if the behavior we observe here is typical.

Analytically continue $a_f(s)$ to $-1/2$ to apply the methods of Dueñez et al. to compute the cutoff in a different way that can be compared to numerical results.

Acknowledgments

- Advisor Steven J Miller
- Professors Zhengyu Mao, Nina Snaith, Gonzalo Tornaría for their helpful insights
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