Torsion points and concurrent exceptional curves on Del Pezzo surfaces of degree 1

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on a joint work with R. Winter

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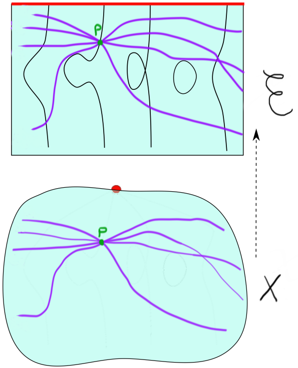
In ultima rex

X del Pezzo of degree 1.
$\mathscr{E}$ rational elliptic surface obtained from X by blowup.
$P \in X$ point at the intersection of many exceptional curves.



Q: When is $P \in X$ a torsion point on its fibre of $\mathscr{E}$?

Plan of the talk

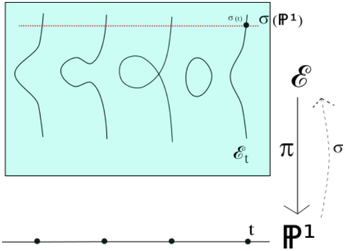
0. In ultima res
1. Misty opening, our protagonists and initial set up
2. Flash back to the inciting incident
3. The resolution and a cliffhanger

1. Set up

Our main protagonist:

An elliptic surface $\mathscr{E}$ with base $\mathbb{P}^1_k$ is:

- a smooth, projective surface
- fibered in elliptic curves:
  - $\pi : \mathscr{E} \longrightarrow \mathbb{P}^1_k$ is such that a fiber $\mathscr{E}_t := \pi^{-1}(t)$ has genus 1 (finitely many exception)
  - there exists a section to π



Equivalently: there exists a Weierstrass equation $y^2 = x^3 + F(T)x + G(T)$, with $F, G \in \mathbb{Q}[T]$, describing the surface.

Misty opening:

Silverman's Specialization Theorem

Let $\mathscr{E}_T$ be an elliptic surface with base $\mathbb{P}^1$ over k extension of $\mathbb{Q}$, then for all $t \in k[T]$ except finitely many:

$$r_{k[T]}(\mathscr{E}_T) \leq r_k(\mathscr{E}_t).$$

1. When do we have a **rank fall**? $t \in k$ such that $r_{k[T]}(E_T) > r_k(E_t)$.
2. When do we have a **rank jump**? $t \in k$ such that $r_{k[T]}(E_T) < r_k(E_t)$.

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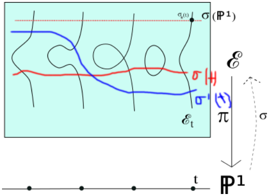

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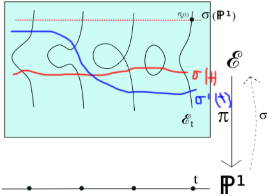

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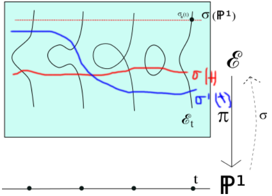

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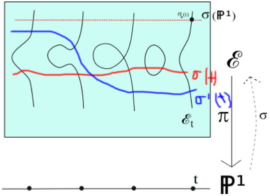

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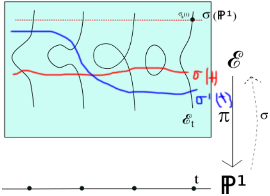

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Warning: there could be non-torsion points on the fibers unrelated to the sections!

Our ally:

- X Del Pezzo surface
  - smooth, projective, geometrically integral over k
  - with ample $-K_X$
  - $1 \leq$degree≤ 9 is $(K_X.K_X)$

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  $0 \leq \deg F \leq 4, 0 \leq \deg G \leq 6$, $1 \leq \deg \Delta \leq 12$
- on a Del Pezzo surface of degree d, blow up $9 - d$ points to obtain a rational elliptic surface.

DP1 $\dashrightarrow \mathscr{E}$

- Let S be a del Pezzo surface of degree one on a field k. Then S is isomorphic to a sextic hypersurface there is an equation of the form
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Example: Isotrivial rational elliptic surfaces: $\mathscr{E} : y^2 = x^3 + \tilde{G}(T)$ where $\tilde{G}(T)$ is squarefree and $\deg \tilde{G} = 5, 6$.

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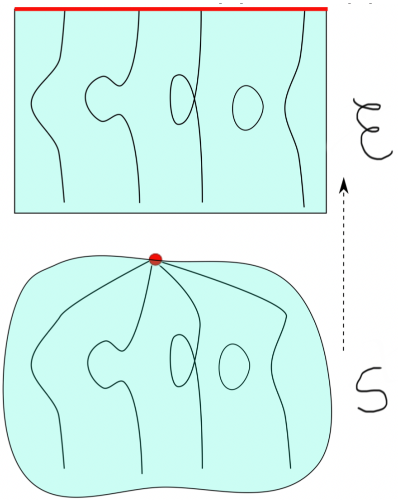

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$$A_1 \times A_2,\ A_4,\ D_5,\ E_6,\ E_7,\ E_8.$$

Thus

$d(X)$	7	6	5	4	3	2	1
# of exceptional curves	3	6	10	16	27	56	240

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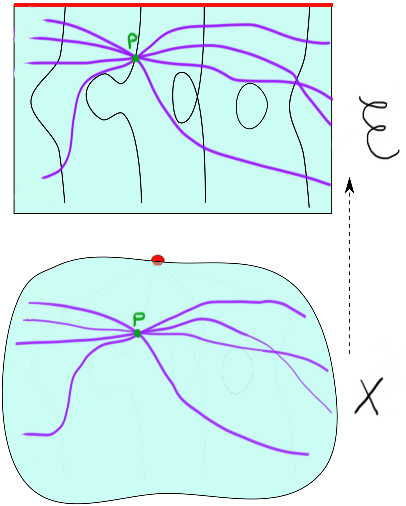

2. Inciting element

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If $k = \mathbb{Q}$, suppose $\mathscr{E}$ is **non-isotrivial**, then under analytical number theory conjectures on certain factors of $\Delta_{\mathscr{E}}$, we have $\#\{t \in \mathbb{Q} \ : \ W(\mathscr{E}_t) = -1\} = \infty$.

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Degree 1? 2?

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Q: What about the del Pezzo surfaces of degree 1 with no conic bundle structure?

Zariski-density of Del Pezzo of degree 1

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Theorem (D.-Winter 2022)

For a certain (isotrivial!) family, the rational points are dense assuming $\exists P \in S$ **non-torsion** on its fiber.

3. Resolution

Initial question

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Theorem (Kuwata 2005)

For del Pezzo surfaces of degree 2, if 'many' equals 4, then yes.

Some answer

Let X be a del Pezzo surface of degree 1, and $\mathscr{E}$ the corresponding elliptic surface.

Theorem

If a point on X is contained in a least 9 exceptional curves, then it is torsion on its fiber on $\mathscr{E}$.

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- The 240 lines on X are sections on $\mathscr{E}$. Those sections generate the group $MW(\mathscr{E})$, which is torsion free and has rank at most 8.
- In $\mathbb{P}^2$, those exceptional curves correspond to:
  - One of the pt P_i
  - A line passing through two of the P_i's
  - A conic passing through five of the P_i's
  - A cubic passing through seven of the P_i's (one double point)
  - A quartic passing through eight of the P_i's (three double points)
  - A quintic passing through eight of the P_i's (6 double points)
  - A sextic through 8 of P_i's (7 double points, 1 triple pt)

Sketch of the proof

If $n \geq 9$ lines intersect in a point $P \in X$, then the correspond to n sections on $\mathscr{E}$, say $S_1, \ldots, S_n$ which all intersect in P.

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If $n \geq 9$ lines intersect in a point $P \in X$, then the correspond to n sections on $\mathscr{E}$, say $S_1, \ldots, S_n$ which all intersect in P.
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Need to show that there is a vector $v \in \ker M$ that does not sum to 0.

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- The isomorphism type of a set of lines determines their Gram matrix.
- (van Luijk - Winter 2021) List of all isomorphism type of maximal cliques in weighted graphs on E_8.
- As a consequence of their work:

Theorem (van Luijk - Winter 2021)

If $\mathrm{char} k = 0$, a point on X is contained in at 10 exceptional curves.

Putting everything together

Let $e_1, \ldots, e_n$ be $n \geq 9$ exceptional curves on X, and assume that they meet in a point $P \in X$.

We want to show that P is torsion on its fiber of $\mathscr{E}$.

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- So 'many'$= 9$ implies yes to the question. What if many is smaller than 9?
- To get all the isomorphism types of intersection graphs of 8 intersecting exceptional curves we need to aditionnally consider:
  - 29 maximal sets of size 8.

Approach for 7 and 8 lines

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- We constructed counter-examples from one of the 13 remaining types.
- Let us construct first a DP1 with a point on 5 exceptional curves that is non-torsion on $\mathscr{E}$, from the type associated to the clique $\{L_{1,2}, L_{3,4}, L_{5,6}, L_{7,8}, C_{1,2}, Q_{2,3,5}, Q_{2,4,7}, Q_{3,6,8}, \}$.
  - $L_{i,j}$ = line through i and j,
  - $C_{i,j}$ = cubic passing through all the points except P_i (P_j double),
  - $Q_{i,j,k}$ = quartic passing through all the points (P_i, P_j, P_k triple).

5 exceptional curves meet at a non-torsion point

We take the following 8 points of $\mathbb{P}^2$:

$$P_1 := [0,1,1]; \quad P_2 := [0,1,a]; \quad P_3 := [1,0,1]; \quad P_4 := [1,0,b];$$
$$P_5 := [1,1,1]; \quad P_6 := [1,1,u]; \quad P_7 := [m,1,v]; \quad P_8 := [m,1,c]$$

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$$\mathscr{E}_0 = x^3 + \frac{404107}{74298}x^2y - \frac{1537}{2562}x^2z - \frac{118214}{12383}xy^2 + \frac{305177}{74298}xyz - \frac{1025}{2562}xz^2 + \frac{28956}{12383}y^3 - \frac{43434}{12383}y^2z + \frac{14478}{12383}yz^2.$$

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(Thanks Magma! $\ddot\smile$)

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Example (Desjardins-Winter)

Let X be the blow-up of $\mathbb{P}^2$ in the eight points:

$$P_1 = [0,1,1] \quad P_2 = [0,3861,1957] \quad P_3 = [1,0,1] \quad P_4 = [1188,0,-19]$$
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Can 8 exceptional curves meet at a non-torsion point?

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To be followed...

Thank you for your attention!