Vanishing of twisted L-functions of elliptic curves over function fields

Joint work with
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Elliptic curves over $\mathbb{Q}$

Let E be an elliptic curve over $\mathbb{Q}$ with conductor N_E and L-function

$$L(E,s) = \prod_{p \nmid N_E} \left(1 - \alpha_p p^{-s}\right)^{-1} \left(1 - \overline{\alpha}_p p^{-s}\right)^{-1} \prod_{p \mid N_E} \left(1 - a_p p^{-s}\right)^{-1},$$

where for each $p \nmid N_E$, $E/\mathbb{F}_p$ is an elliptic curve with $p + 1 - (\alpha_p + \overline{\alpha}_p)$ points, with $|\alpha_p| = \sqrt{p}$, and $|a_p| = |\alpha_p + \overline{\alpha}_p| \leq 2\sqrt{p}$ (Hasse bound).

$L(E,s)$ satisfies the functional equation (Wiles, 1995)

$$\Lambda(E,s) = \left(\frac{\sqrt{N_E}}{2\pi}\right)^s \Gamma(s) L(E,s) = \omega_E \, \Lambda(E, 2-s), \;\; \omega_E = \pm 1.$$

Vanishing at $s = 1$ is related to the rational solutions of $E/\mathbb{Q}$ via the **Birch and Swinnerton-Dyer conjecture:**

$$\mathrm{ord}_{s=1} L(E,s) = \mathrm{rank}(E(\mathbb{Q})).$$

The order of vanishing of $L(E,s)$ at $s = 1$ is the analytic rank of E.

Twisted L-functions of elliptic curves

Let χ be a Dirichlet character over $\mathbb{Q}$, and consider the twisted L-function

$$L(E,\chi,s) = \prod_{p\nmid N_E} \left(1-\chi(p)\alpha_p p^{-s}\right)^{-1} \left(1-\chi(p)\overline{\alpha}_p p^{-s}\right)^{-1} \times \prod_{p|N_E} \times \left(1-\chi(p)a_p p^{-s}\right)^{-1} = \sum_n a_n \chi(n) n^{-s}.$$

Suppose that χ has prime order ℓ, and $K/\mathbb{Q}$ is the cyclic extension of order ℓ associated with χ. Then,

$$\zeta_K(s) = \zeta(s) \prod_{\substack{\psi\in\widehat{\mathrm{Gal}(K/\mathbb{Q})}\\ \psi\neq\psi_0}} L(s,\psi) = \zeta(s)\prod_{j=1}^{\ell-1} L(s,\chi^j).$$

Also,

$$L(E/K,s) = L(E,s)\prod_{j=1}^{\ell-1} L(E,\chi^j,s).$$

Rank growth in cyclic extensions of order $\ell \geq 3$

Heuristics based on the distribution of modular symbols and random matrix theory (D-Fearnley-Kisilevsky 2007 and Mazur-Rubin 2015) predict that the vanishing of $L(E, \chi, s)$ at $s = 1$ is a very rare event as χ ranges over characters of prime order $\ell \geq 3$.

Diophantine Stability: Mazur-Rubin (2020) conjectured that if $K/\mathbb{Q}$ is an abelian extension such that K contains only finitely many subfields of degree $2, 3, 5$ over $\mathbb{Q}$, then $E(K)$ is finitely generated.

Larsen-Mazur-Rubin (2018) proved that for a positive proportion of primes ℓ, there are infinitely many ℓ-cyclic extensions $K/\mathbb{Q}$ of order ℓ such that $E(K) = E(\mathbb{Q})$.

Rank growth in quadratic extensions

For **quadratic twists**, $L(E, \chi_D, s) = L(E_D, s)$ where

$$E : y^2 = x^3 + Ax + B, \quad E_D : Dy^2 = x^3 + Ax + B.$$

If $D \nmid N_E$,

$$\omega_{E_D} = \omega_E \, \chi_D(N_E),$$

and for half of the quadratic twists, $L(E, \chi_D, 1) = L(E_D, 1) = 0$.

Goldfeld (1974) conjectured that half of the twists $E_D/\mathbb{Q}$ have analytic rank zero, and half have analytic rank one (asymptotically).

- Heath-Brown (2004): a positive proportion of the twists have analytic rank 0 and analytic rank 1 (assuming GRH).
- Smith (2022): almost all elliptic curves satisfy Goldfeld conjecture (assuming BSD), generalizing Heath-Brown (1994).
- Gouvea and Mazur (1991): the analytic rank of E_D is at least two for $\gg X^{1/2-\epsilon}$ discriminants $|D| \leq X$.

Rank growth in non-abelian extensions

For the case of extensions $K/\mathbb{Q}$ of degree d with Galois group S_d, Lemke Oliver and Thorne (2021) showed that there are infinitely many such extensions where $\mathrm{rank}(E(K)) > \mathrm{rank}(E(\mathbb{Q}))$, for each $d \geq 2$.

Fornea (2019) has shown that for some curves $E/\mathbb{Q}$, the analytic rank of E increases for a positive proportion of the quintic fields with Galois group S_5.

Under certain conditions on E, Keliher (2022) has showed that there are infinitely many $K/\mathbb{Q}$ with Galois group S_4 such that the rank does not increase.

Number fields and function fields

Let q power of a prime, $\mathbb{F}_q$ finite field with q elements.

Number Fields		Function Fields
$\mathbb{Q}$	$\leftrightarrow$	$\mathbb{F}_q(T)$
$\mathbb{Z}$	$\leftrightarrow$	$\mathbb{F}_q[T]$
p positive prime	$\leftrightarrow$	$P(T)$ irreducible polynomial, or P_∞
$\|n\| = \|\mathbb{Z}/n\mathbb{Z}\| = n$	$\leftrightarrow$	$\|F(T)\| = \|\mathbb{F}_q[T]/(F(T))\| = q^{\deg F}$
$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$	$\leftrightarrow$	$\zeta_q(s) = \sum_{\substack{F \in \mathbb{F}_q[T] \\ F \text{ monic}}} \frac{1}{\|F\|^s} = \frac{1}{1 - q^{s-1}}$
Riemann Hypothesis ???	$\leftrightarrow$	Riemann Hypothesis !!!

Vanishing of L-functions over function fields

Before considering the L-functions $\mathcal{L}(E, \chi, u)$, let's discuss the L-functions $\mathcal{L}(\chi, u)$. We use the change of variable $u = q^{-s}$.

Let χ be a Dirichlet character of order ℓ over $\mathbb{F}_q(t)$ with conductor $F_\chi \in \mathbb{F}_q[t]$ and L-function

$$\mathcal{L}(\chi, u) = \prod_P (1 - \chi(P) u^{\deg P})^{-1} \;\; (P \in \mathbb{F}_q[t] \text{ or } P = P_\infty).$$

It follows from the work of Weil that $\mathcal{L}(\chi, u)$ is a polynomial in u of degree $\deg F_\chi - 2 + \delta_\chi$, and

$$\mathcal{L}(\chi, u) = \prod_{j=1}^{\deg F_\chi - 2 + \delta_\chi} \left(1 - u q^{1/2} e^{i\theta_j}\right).$$

Furthermore, $\mathcal{L}(\chi, u)$ satisfies the functional equation

$$\mathcal{L}(\chi, u) = \omega_\chi \, (\sqrt{q} u)^{\deg F_\chi - 2 + \delta_\chi} \, \mathcal{L}(\overline{\chi}, 1/qu),$$

relating u to $1/qu$ and then s to $1 - s$.

Vanishing of L-functions over function fields

The vanishing of $\mathcal{L}(\chi, u)$ at $u = q^{-\frac{1}{2}}$ correspond to vanishing of $L(s, \chi)$ at $s = \frac{1}{2}$. Over $\mathbb{Q}$, Chowla conjectured that $L(\frac{1}{2}, \chi) \neq 0$.

Theorem (Li 2018, Donepudi-Li 2021)

- *There are at least $\gg q^{\frac{n}{3}-\varepsilon}$ of the Cq^n quadratic characters of conductor of degree bounded by n such that $\mathcal{L}(\chi, q^{-\frac{1}{2}}) = 0$. If $q = p^{2e}$, this can be improved to $\gg q^{\frac{n}{2}-\varepsilon}$.*
- *If $q = p^{4e}$, there are at least $q^{\frac{2n}{3}-\varepsilon}$ of the $\approx q^n$ cubic characters of conductor of degree bounded by n such that $\mathcal{L}(\chi, q^{-\frac{1}{2}}) = 0$.*
- *If $p \equiv -1 \bmod \ell$ and $q = p^d$ for d large enough, there are at least $q^{\frac{2n}{3}-\varepsilon}$ of the $\approx q^n$ characters of order ℓ of conductor of degree bounded by n such that $\mathcal{L}(\chi, q^{-\frac{1}{2}}) = 0$.*

L-functions of elliptic curves over function fields

Let E be an elliptic curve over $\mathbb{F}_q(t)$, say

$$E : y^2 = x^3 + Ax + B, \quad A, B \in \mathbb{F}_q[t].$$

Let P be a prime of $\mathbb{F}_q(t)$ (including $P = P_\infty$), and

$$\mathbb{F}_P = \mathbb{F}_q[t]/(P) \cong \mathbb{F}_{q^{\deg P}}.$$

If P is a prime of good reduction $(P \nmid N_E)$

$$\#E(\mathbb{F}_P) = q^{\deg P} + 1 - a_P, \quad a_P = \alpha_P + \overline{\alpha}_P, \quad |\alpha_P| = \sqrt{q^{\deg P}}.$$

Let

$$\mathcal{L}_P(E, u) = 1 - a_P u + q^{\deg P} u^2 = (1 - \alpha_P u)(1 - \overline{\alpha}_P u)$$

be the L-function of $E/\mathbb{F}_P$.

L-functions of elliptic curves over function fields

The L-function of $E/\mathbb{F}_q(t)$ is defined by

$$\mathcal{L}(E,u) = \prod_{P \nmid N_E} \mathcal{L}_P(E, u^{\deg P})^{-1} \prod_{P | N_E} \mathcal{L}_P(E, u^{\deg P})^{-1}.$$

From Deligne (1981), if E a non-constant elliptic curve over $\mathbb{F}_q(t)$, $\mathcal{L}(E,u)$ is a polynomial of degree $\deg N_E - 4$ and

$$\mathcal{L}(E,u) = \prod_{j=1}^{\deg N_E - 4} \left(1 - que^{i\theta_j}\right).$$

Then, $\mathcal{L}(E,u)$ satisfies the functional equation

$$\mathcal{L}(E,u) = \omega_E \, (qu)^{\deg(N_E)-4} \mathcal{L}(E, 1/(q^2 u)), \;\; \omega_E = \pm 1,$$

relating u to $1/q^2 u$ and then s to $2-s$.

This comes from seeing $E/\mathbb{F}_q(t)$ as a surface over $\mathbb{F}_q$.

Elliptic curves L-functions twisted by Dirichlet characters

The twisted L-function $\mathcal{L}(E,\chi,u)$ is defined by

$$\prod_{P\nmid N_E}(1-\chi(P)\alpha_P u^{\deg(P)})^{-1}(1-\chi(P)\overline{\alpha}_P u^{\deg(P)})^{-1} \times \prod_{P|N_E}(1-\chi(P)a_P u^{\deg(P)})^{-1}.$$

If $(N_E, F_\chi)=1$, then $\mathcal{L}(E,\chi,u)$ is a polynomial of degree

$$N := \deg N_E + 2\deg F_\chi - 4 + 2\delta_\chi$$

and satisfy the functional equation

$$\mathcal{L}(E,\chi,u) = \omega_{E\otimes\chi}\,(qu)^N\,\mathcal{L}(E,\overline{\chi},1/(q^2u)), \quad \omega_{E\otimes\chi} = \omega_\chi^2\,\omega_E\,\chi(N_E).$$

If $K/\mathbb{F}_q(t)$ is the cyclic extension of order ℓ associated to χ,

$$\mathcal{L}(E/K,u) = \mathcal{L}(E,u)\prod_{i=1}^{\ell-1}\mathcal{L}(E,\chi^i,u).$$

Characters, extensions $K/\mathbb{F}_q(t)$ and curves over $\mathbb{F}_q$

We can associate characters $\chi/\mathbb{F}_q(t)$ with abelian extensions $K/\mathbb{F}_q(t)$.
We can associate extensions $K/\mathbb{F}_q(t)$ with curves $C/\mathbb{F}_q$ by $K = \mathbb{F}_q(C)$, the function field of C.

For example, let C be the hyperelliptic curve $C : y^2 = f(t)$, $f(t) \in \mathbb{F}_q(t)$. Then,

$$\mathbb{F}_q(C) = \mathbb{F}_q[t, y]/(y^2 - f(t)) = \mathbb{F}_q(t)(\sqrt{f(t)})$$

is a quadratic extension.

Characters of order ℓ are associated with ℓ-cyclic extensions K, which are associated with ℓ-cyclic covers C, for example

$$C : y^\ell = f(t) \quad (q \equiv 1 \bmod \ell),$$

and we have

$$\mathcal{L}(C, u) = \prod_{i=1}^{\ell-1} \mathcal{L}(\chi^i, u), \qquad \mathcal{Z}(C, u) = \mathcal{Z}(u)\mathcal{L}(C, u).$$

Constant elliptic curves over $\mathbb{F}_q(t)$

Let C be a ℓ-cyclic cover over $\mathbb{F}_q$ of genus g and L-function

$$\mathcal{L}(C,u) = \prod_{j=1}^{2g}(1-\beta_j u), \ \ |\beta_j| = q^{1/2}.$$

Let E_0 be an elliptic curve over $\mathbb{F}_q$ with L-function

$$\mathcal{L}(E_0,u) = (1-\alpha_1 u)(1-\alpha_2 u), \ \ |\alpha_1|, |\alpha_2| = q^{1/2}.$$

Then $E = E_0 \times_{\mathbb{F}_q} \mathbb{F}_q(t)$ is a constant elliptic curve over $\mathbb{F}_q(t)$.

Theorem (Milne, 1968)

$$\begin{aligned}\mathcal{L}(E/K_C,u) &= \mathcal{Z}(C,\alpha_1 u)\mathcal{Z}(C,\alpha_2 u) \\ &= \frac{\prod_{\substack{1\le i\le 2\\ 1\le j\le 2g}}(1-\alpha_i\beta_j u)}{\prod_{1\le i\le 2}(1-\alpha_i u)(1-\alpha_i q u)}.\end{aligned}$$

L-functions of constant elliptic curves

Corollary

If $E = E_0 \times_{\mathbb{F}_q} \mathbb{F}_q(t)$, $\mathcal{L}(E/K_C, q^{-1}) = 0$ *if and only if* $\mathcal{L}(C, \alpha_1^{-1}) = \mathcal{L}(C, \alpha_2^{-1}) = 0$.

We recall that

$$\mathcal{L}(C, u) = \prod_{j=1}^{2g}(1 - \beta_j u) = \prod_{i=1}^{\ell-1} \mathcal{L}(\chi^i, u),$$

$$\mathcal{L}(E/K_C, u) = \mathcal{L}(E, u) \prod_{i=1}^{\ell-1} \mathcal{L}(E, \chi^i, u).$$

The vanishing of $\mathcal{L}(E, \chi, u)$ at $u = q^{-1}$ reduces to: The vanishing of $\mathcal{L}(\chi, u)$ at $u = \alpha^{-1}$ where $\mathcal{L}(E_0, u) = (1 - \alpha)(1 - \overline{\alpha})$.

Vanishing for constant elliptic curves

We generalize the work of Donepudi-Li to general ℓ-cyclic cover $C/\mathbb{F}_q$ (and not only the Kummer ones where $q \equiv 1 \mod \ell$), using the work of Bary-Soroker and Meisner (2019).

Theorem

If there is one ℓ-cyclic cover $C_0/\mathbb{F}_q$ such that $\mathcal{L}(C_0, u_0) = 0$, then there are at least q^{2n/d_0} of the $\approx q^n$ ℓ-cyclic cover $C/\mathbb{F}_q$ with conductor of degree bounded by n such that $\mathcal{L}(C, u_0) = 0$, where d_0 is the degree of the conductor of C_0.

Theorem

Let E_0 be an elliptic curve over $\mathbb{F}_q$, and let $E = E_0 \times_{\mathbb{F}_q} \mathbb{F}_q(t)$. If there is one Dirichlet character χ_0 of order ℓ over $\mathbb{F}_q(t)$ with conductor of degree d_0 such that $\mathcal{L}(E, \chi_0, q^{-1}) = 0$, there are at least $\gg q^{2n/d_0}$ Dirichlet characters of order ℓ over $\mathbb{F}_q(t)$ with conductor of degree bounded by n such that $\mathcal{L}(E, \chi, q^{-1}) = 0$.

Geometric vanishing criterion

Tate-Honda theory: There is a one-to-one correspondance between conjucacy classes of q-Weil numbers and isogeny classes of simple Abelian varieties over $\mathbb{F}_q$. Furthermore, B is isogenous to a sub-abelian variety of A if and only if $P_B(x)$ divides $P_A(x)$, where $P_A(x)$ is the characteristic polynomial of Frobenius.

Theorem (Li, 2018)

Let u_0 be a q-Weil number and let A_0 be the unique (isogeny class of) simple Abelian variety over $\mathbb{F}_q$ having u_0 as a Frobenius eigenvalue, as guaranteed by the theorem of Honda–Tate.

Suppose that $A_0 = Jac(C_0)$ for some curve $C_0/\mathbb{F}_q$. Let C be a curve over $\mathbb{F}_q$.

Then, $\mathcal{L}(C, u_0^{-1}) = 0$ if and only if there exists a non-trivial map $C \to C_0$ if and only if $\mathcal{L}(C_0, u)$ divides $\mathcal{L}(C, u)$.

Kummer ℓ-cyclic covers

If $q \equiv 1 \bmod \ell$, let C_0 be the ℓ-cyclic cover

$$C_0 : y^\ell = f(t),\ f(t) = f_1 f_2^2 \dots f_{\ell-1}^{\ell-1},$$

where $F_0 = f_1 f_2 \dots f_{\ell-1}$ is square-free and $d_0 := \deg(f_1 \dots f_{\ell-1})$.

Let $h(t) \in \mathbb{F}_q[t]$, and let C be the curve

$$C : y^\ell = f(h(t)),$$

where $F(t) = F_0(h(t)) = f_1(h(t)) f_2(h(t)) \dots f_{\ell-1}(h(t))$ is square-free, and $\deg F = d_0 \cdot \deg h$.

There is a non-trivial map of curves

$$\begin{array}{rccc} \phi : & C & \rightarrow & C_0 \\ & (t, y) & \mapsto & (h(t), y) \end{array}$$

and we have to count the square-free values $(f_1 \dots f_{\ell-1})(h(t))$.

Square-free values of polynomials over $\mathbb{F}_q[t]$

Proposition (Poonen, 2003)

Let $f \in \mathbb{F}_q[t][x_1, \ldots, x_m]$ be a polynomial that is square-free as an element of $\mathbb{F}_q(t)[x_1, \ldots, x_m]$. Let

$$S_f := \{a \in \mathbb{F}_q[t]^m : f(a) \text{ is square-free}\},$$

$$||a|| := \max_{1 \leq i \leq m} |a_i|$$

$$\mu(S_f) := \lim_{N \to \infty} \frac{|\{a \in S_f : ||a|| < N\}|}{|\{a \in \mathbb{F}_q[t]^m : ||a|| < N\}|}$$

For each nonzero prime $\mathfrak{p}$ of $\mathbb{F}_q[t]$, let $c_{\mathfrak{p}}$ be the number of $x \in (\mathbb{F}_q[t]/\mathfrak{p}^2)^m$ that satisfy $f(x) = 0$ in $A/\mathfrak{p}^2$. The limit $\mu(S_f)$ exists and is equal to $\prod_{\pi}(1 - c_{\mathfrak{p}}/|\mathfrak{p}|^{2m})$.

For square-free values of polynomials over $\mathbb{Z}$, Poonen proved the same result assuming the abc-conjecture, following the work of Granville (1998).

General ℓ-cyclic covers

Let n_q be the multiplicative order of $q \bmod \ell$.

The conductors of characters of order ℓ are square-free $F \in \mathbb{F}_q[t]$ supported on prime polynomials $P \in \mathbb{F}_q[t]$ with $n_q \mid \deg P$, or equivalently, P which split completely in $\mathbb{F}_{q^{n_q}}(t)/\mathbb{F}_q(t)$.

Writing F as a product of n_q conjugates in $\mathbb{F}_{q^{n_q}}(t)$,

$$F = \mathfrak{F}_1 \dots \mathfrak{F}_{n_q},\ \phi_q(\mathfrak{F}_i) = \mathfrak{F}_{i+1} \Rightarrow N_q(\mathfrak{F}_i) = F,$$

one can write the equation of C in terms of $\mathfrak{F}_1, \dots \mathfrak{F}_{n_q}$ (Bary-Soroker-Meisner, 2019).

For example, if $\ell = 3$ and $q \equiv 2 \bmod 3$ (and $n_q = 2$)

$$C_F \ : \ y^3 - 3\mathfrak{F}_1\mathfrak{F}_2 y - \mathfrak{F}_1\mathfrak{F}_2(\mathfrak{F}_1 + \mathfrak{F}_2) = 0.$$

General ℓ-cyclic covers

For general ℓ-cyclic covers, using the result of Poonen about square-free values of multi-variable polynomials over $\mathbb{F}_q[t]$, we can show that

Theorem

Let E_0 be an elliptic curve over $\mathbb{F}_q$, and let $E = E_0 \times_{\mathbb{F}_q} \mathbb{F}_q(t)$. If there is one Dirichlet character χ_0 of order ℓ over $\mathbb{F}_q(t)$ with conductor of degree d_0 such that $\mathcal{L}(E, \chi_0, q^{-1}) = 0$, there are at least $\gg q^{2n/d_0}$ Dirichlet characters of order ℓ over $\mathbb{F}_q(t)$ with conductor of degree bounded by n such that $\mathcal{L}(E, \chi, q^{-1}) = 0$.

Remark: The d_0 comes from the initial curve C_0. The 2 comes from the fact that we use $h(t) = u(t)/v(t)$ in the map from C to C_0 by homegenizing the equations, so we use Poonen's sieve with $m = 2$ for the tuples $(u(t), v(t))$.

Let $E = E_0 \times_{\mathbb{F}_q} \mathbb{F}_q(t)$. How do you produce a ℓ-cyclic cover C_0 over $\mathbb{F}_q$ such that $\mathcal{L}(E/K_{C_0}, q^{-1}) = 0$?

Examples for constant curves

- Over $\mathbb{F}_{13}$, the 7-cyclic cover C_0 ($n_q = 2$) with equation

$$y^7 + (6t^4 + 6t^3 + 6t^2 + 12t + 1)y^5 +$$
$$(t^8 + 2t^7 + 3t^6 + 6t^5 + t^4 + 5t + 4)y^3 +$$
$$(6t^{12} + 5t^{11} + 10t^{10} + 7t^8 + 2t^7 + \cdots + 2t^3 + 6t^2 + t + 4)y +$$
$$11t^{14} + 6t^{13} + 12t^{12} + 10t^{11} + \cdots + 7t^4 + 12t^3 + 3t^2 + 3t + 9 = 0$$

has $\mathcal{L}(C_0, u) = (1 + 13u^2)^6 = \mathcal{L}(E_0, u)^6$ where E_0 is a supersingular elliptic curve over $\mathbb{F}_{13}$. Then, $\mathcal{L}(C_0, \alpha_0) = 0$ and $\mathcal{L}(E/K_{C_0}, q^{-1}) = 0$ for $E = E_0 \times_{\mathbb{F}_q} \mathbb{F}_q(t)$, and there are infinitely many such 7-cyclic covers C.

- Let $E_0/\mathbb{F}_{73}$ be a supersingular elliptic curve, and $E = E_0 \times_{\mathbb{F}_q} \mathbb{F}_q(t)$. There is a character χ_0 of order 37 over $\mathbb{F}_{73}[t]$ such that $\mathcal{L}(E, \chi, q^{-1}) = 0$, and there are infinitely many such χ of order 37.

Numerical data for non-constant curves

We also computed numerically $\mathcal{L}(E, \chi, u)$ for non-constant E and χ of order ℓ. Let F_χ be the conductor of χ. The twisted L-functions are polynomials of degree $N = \deg N_E + 2 \deg F_\chi - 4 + 2\delta_\chi$

$$\mathcal{L}(E, \chi, u) = \sum_{n=0}^{N} \left(\sum_{f \in \mathcal{M}_n} a_f \chi(f) \right) u^n = \sum_{n=0}^{N} c_n u^n.$$

Using the functional equation,

$$c_n = \omega_{E \otimes \chi} \, p^{2(n - \lfloor N/2 \rfloor - 1)} \, \overline{c_{N-n}}, \;\; 0 \le n \le N,$$

and it suffices to compute c_i for $0 \le i \le \lfloor N/2 \rfloor$.

The next slide shows data for the analytic rank of $\mathcal{L}(E, \chi, u)$ at $u = q^{-1}$, for twists of order 3 and

$$E : y^2 = (x-1)(x - 2t^2 - 1)(x - t^2).$$

Numerical data for non-constant curves

p	n_p	deg(conductor χ)	rank 0	rank 1	rank 2	rank 3
5	2	2	8	2	0	0
		4	214	26	0	0
		6	5780	280	0	0
		8	149222	2136	20	2
7	1	1	4	0	0	0
		2	30	2	0	0
		3	264	22	2	0
		4	2299	49	4	0
		5	18670	240	2	0
		6	148537	1343	32	0