Exceptional Howe correspondences and Arthur packets for G_2

Petar Bakić, University of Utah

joint w/ G. Savin

Québec, 16 October 2022

Contents

Preliminaries

Preliminaries

$\mathbb{F}$ = non-Archimedean local field, char 0

Preliminaries

$\mathbb{F}$ = non-Archimedean local field, char 0

Groups: $G(\mathbb{F})$, where G is an algebraic group

Preliminaries

$\mathbb{F}$ = non-Archimedean local field, char 0

Groups: $G(\mathbb{F})$, where G is an algebraic group

Representations: complex, smooth

The classical theta correspondence

W = symplectic space (over $\mathbb{F}$)
$\mathrm{Sp}(W)$ = group of isometries of the symplectic form on W.

$\mathrm{Mp}(W)$ = metaplectic group

The classical theta correspondence

W = symplectic space (over $\mathbb{F}$)
$\mathrm{Sp}(W)$ = group of isometries of the symplectic form on W.

$\mathrm{Mp}(W)$ = metaplectic group

Key ingredients for theta correspondence:

The classical theta correspondence

W = symplectic space (over $\mathbb{F}$)
$\mathrm{Sp}(W)$ = group of isometries of the symplectic form on W.

$\mathrm{Mp}(W)$ = metaplectic group

Key ingredients for theta correspondence:

- Weil representation ω of $\mathrm{Mp}(W)$

The classical theta correspondence

W = symplectic space (over $\mathbb{F}$)
$\mathrm{Sp}(W)$ = group of isometries of the symplectic form on W.

$\mathrm{Mp}(W)$ = metaplectic group

Key ingredients for theta correspondence:

- Weil representation ω of $\mathrm{Mp}(W)$
- Dual pair $G \times G' \hookrightarrow \mathrm{Mp}(W)$

$$G \xleftrightarrow{\text{centralizer}} G'$$

The classical theta correspondence

W = symplectic space (over $\mathbb{F}$)
$\mathrm{Sp}(W)$ = group of isometries of the symplectic form on W.

$\mathrm{Mp}(W)$ = metaplectic group

Key ingredients for theta correspondence:

- Weil representation ω of $\mathrm{Mp}(W)$
- Dual pair $G \times G' \hookrightarrow \mathrm{Mp}(W)$

$$G \xleftrightarrow{\text{centralizer}} G'$$

Problem: Describe the restriction of ω to $G \times G'$.

Fix $\pi \in \mathrm{Irr}(G)$. The maximal π-isotypic quotient of ω is of the form

$$\pi \otimes \Theta(\pi),$$

where $\Theta(\pi)$ is a smooth representation of G'.

Fix $\pi \in \mathrm{Irr}(G)$. The maximal π-isotypic quotient of ω is of the form

$$\pi \otimes \Theta(\pi),$$

where $\Theta(\pi)$ is a smooth representation of G'.

Theorem (Howe duality)

- $\Theta(\pi)$ is either 0 or an admissible representation of finite length.
- If non-zero, $\Theta(\pi)$ has a unique irreducible quotient, $\theta(\pi)$.
- Injectivity:
$$\theta(\pi) \cong \theta(\pi') \neq 0 \Rightarrow \pi \cong \pi'.$$

Fix $\pi \in \mathrm{Irr}(G)$. The maximal π-isotypic quotient of ω is of the form

$$\pi \otimes \Theta(\pi),$$

where $\Theta(\pi)$ is a smooth representation of G'.

Theorem (Howe duality)

- $\Theta(\pi)$ is either 0 or an admissible representation of finite length.
- If non-zero, $\Theta(\pi)$ has a unique irreducible quotient, $\theta(\pi)$.
- Injectivity:
$$\theta(\pi) \cong \theta(\pi') \neq 0 \Rightarrow \pi \cong \pi'.$$

Correspondence: $\pi \leftrightarrow \theta(\pi)$.

Exceptional theta

We'd like to look at similar constructions in other groups.
We need:

Exceptional theta

We'd like to look at similar constructions in other groups.
We need:

- dual pairs

Exceptional theta

We'd like to look at similar constructions in other groups.
We need:

- dual pairs
- a suitable replacement for the Weil representation

Exceptional dual pairs: an example

Exceptional dual pairs: an example

$\mathbb{O}$ = algebra of octonions over F
$J = 3 \times 3$ hermitian matrices with octonion entries
J comes equipped with det.

Exceptional dual pairs: an example

$\mathbb{O}$ = algebra of octonions over F
$J = 3 \times 3$ hermitian matrices with octonion entries
J comes equipped with det.

The isogenies of det form a (split) reductive group of type E_6.
Inside E_6, we have

Exceptional dual pairs: an example

$\mathbb{O}$ = algebra of octonions over F
$J = 3 \times 3$ hermitian matrices with octonion entries
J comes equipped with det.

The isogenies of det form a (split) reductive group of type E_6.
Inside E_6, we have

G_2, the automorphism group of $\mathbb{O}$

Exceptional dual pairs: an example

$\mathbb{O}$ = algebra of octonions over F
$J = 3 \times 3$ hermitian matrices with octonion entries
J comes equipped with det.

The isogenies of det form a (split) reductive group of type E_6.
Inside E_6, we have

- G_2, the automorphism group of $\mathbb{O}$
- PGL_3, acting on J by conjugation

Exceptional dual pairs: an example

$\mathbb{O}$ = algebra of octonions over F
$J = 3 \times 3$ hermitian matrices with octonion entries
J comes equipped with det.

The isogenies of det form a (split) reductive group of type E_6.
Inside E_6, we have

- G_2, the automorphism group of $\mathbb{O}$
- PGL_3, acting on J by conjugation

We get a dual pair $G_2 \times \mathrm{PGL}_3 \subset E_6$

Exceptional dual pairs: an example

$\mathbb{O}$ = algebra of octonions over F
$J = 3 \times 3$ hermitian matrices with octonion entries
J comes equipped with det.

The isogenies of det form a (split) reductive group of type E_6.
Inside E_6, we have

G_2, the automorphism group of $\mathbb{O}$

PGL_3, acting on J by conjugation

We get a dual pair $G_2 \times \mathrm{PGL}_3 \subset E_6$

We consider the quasi split version: dual pair $G_2 \times \mathrm{PU}_3 \subset E_{6,4}$.

A replacement for Weil?

A replacement for Weil?

The minimal representation.

Theta lifts

Let Π be the minimal representation of H. Restrict Π to the dual pair $G \times G' \subset H$. Given $\pi \in \mathrm{Irr}(G)$, the maximum π-isotypic quotient of Π is of the form

$$\pi \otimes \Theta(\pi).$$

We would like to prove Howe duality:

Theta lifts

Let Π be the minimal representation of H. Restrict Π to the dual pair $G \times G' \subset H$. Given $\pi \in \mathrm{Irr}(G)$, the maximum π-isotypic quotient of Π is of the form

$$\pi \otimes \Theta(\pi).$$

We would like to prove Howe duality:

- $\Theta(\pi)$ is either 0 or a representation of finite length
- If non-zero, $\Theta(\pi)$ has a unique irreducible quotient, $\theta(\pi)$.
- Injectivity: for $\pi, \pi' \in \mathrm{Irr}(G)$ we have

$$\theta(\pi) \cong \theta(\pi') \neq 0 \Rightarrow \pi \cong \pi'.$$

Howe duality — proof strategy

Main ideas:

(1) describe Jacquet modules of minimal rep

(2) play period ping-pong

Periods

Consider the theta correspondence for the dual pair $G \times G'$. Let
π = a representation of G
H = a subgroup of G
χ = a character of H

We refer to

$$\mathrm{Hom}_H(\pi, \chi)$$

as a χ-period of π.

Periods

Consider the theta correspondence for the dual pair $G \times G'$. Let
π = a representation of G
H = a subgroup of G
χ = a character of H

We refer to

$$\mathrm{Hom}_H(\pi, \chi)$$

as a χ-period of π.

General principle: theta correspondence transfers a period on G to a period on G' (and vice versa).

Coinvariants

$B = TU =$ Borel in G_2.

Consider ψ_U-coinvariants of minimal rep:

$$\Pi_{U,\psi_U} \cong \text{c-ind}_S^{G'} \psi_S.$$

$S =$ Shalika subgroup of $G' = \mathrm{PU}_3(K)$

Coinvariants

$B = TU =$ Borel in G_2.

Consider ψ_U-coinvariants of minimal rep:

$$\Pi_{U,\psi_U} \cong \text{c-ind}_S^{G'} \psi_S.$$

$S =$ Shalika subgroup of $G' = \mathrm{PU}_3(K)$

We also look at Π_{S,ψ_S}. Crucial:

Coinvariants

$B = TU =$ Borel in G_2.

Consider ψ_U-coinvariants of minimal rep:

$$\Pi_{U,\psi_U} \cong \text{c-ind}_S^{G'} \psi_S.$$

$S =$ Shalika subgroup of $G' = \mathrm{PU}_3(K)$

We also look at Π_{S,ψ_S}. Crucial:

For $\pi \in \mathrm{Irr}(G)$ we have

$$\dim \mathrm{Hom}_G(\Pi_{S,\psi_S}, \pi) \leqslant 1.$$

Ping-pong

Let $\pi \in \mathrm{Irr}(G)$, $\tau \in \mathrm{Irr}(G')$ be tempered such that

$$\mathrm{Hom}_{G \times G'}(\Pi, \pi \boxtimes \tau) \neq 0.$$

Then

$$\begin{aligned}\mathrm{Hom}_U(\pi, \psi_U) &\overset{(1)}{\subseteq} \mathrm{Hom}_U(\Theta(\tau), \psi_U) \overset{(2)}{\cong} \mathrm{Hom}_S(\tau^\vee, \overline{\psi}_S) \\ &\overset{(3)}{\subseteq} \mathrm{Hom}_S(\Theta(\pi^\vee), \overline{\psi}_S) \overset{(4)}{\cong} \mathrm{Hom}_G(\Pi_{S,\overline{\psi}_S}, \pi^\vee).\end{aligned}$$

If π is generic, then all of the above spaces are one-dimensional.

Proposition

Let $\pi \in \mathrm{Irr}(G)$ be tempered and generic. Then $\Theta(\pi)$ cannot have two irreducible tempered quotients. In particular, the cuspidal part of $\Theta(\pi)$ is either 0, or irreducible.

Proposition

Let $\pi \in \mathrm{Irr}(G)$ be tempered and generic. Then $\Theta(\pi)$ cannot have two irreducible tempered quotients. In particular, the cuspidal part of $\Theta(\pi)$ is either 0, or irreducible.

Proof.

Let τ_1, τ_2 be irreducible and tempered, such that $\Theta(\pi) \twoheadrightarrow \tau_1 \oplus \tau_2$. Then

$$1 = \dim \mathrm{Hom}_S(\tau_1, \psi_S) = \dim \mathrm{Hom}_S(\Theta(\pi), \psi_S) = \dim \mathrm{Hom}_S(\tau_2, \psi_S).$$

□

$$
\begin{aligned}
\mathsf{Hom}_U(\pi,\psi_U) &\overset{(1)}{\subseteq} \mathsf{Hom}_U(\Theta(\tau),\psi_U) \overset{(2)}{\cong} \mathsf{Hom}_S(\tau^\vee,\overline{\psi}_S) \\
&\overset{(3)}{\subseteq} \mathsf{Hom}_S(\Theta(\pi^\vee),\overline{\psi}_S) \overset{(4)}{\cong} \mathsf{Hom}_G(\Pi_{S,\overline{\psi}_S},\pi^\vee).
\end{aligned}
$$

Proposition

Let $\pi \in \mathrm{Irr}(G)$ be tempered and generic. Then $\Theta(\pi)$ cannot have two irreducible tempered quotients. In particular, the cuspidal part of $\Theta(\pi)$ is either 0, or irreducible.

Proof.

Let τ_1, τ_2 be irreducible and tempered, such that $\Theta(\pi) \twoheadrightarrow \tau_1 \oplus \tau_2$. Then

$$1 = \dim \mathrm{Hom}_S(\tau_1, \psi_S) = \dim \mathrm{Hom}_S(\Theta(\pi), \psi_S) = \dim \mathrm{Hom}_S(\tau_2, \psi_S).$$

But $\tau_1 \oplus \tau_2$ is a quotient of $\Theta(\pi)$, so

$$1 = \dim \mathrm{Hom}_S(\Theta(\pi), \psi_S) \geqslant \dim \mathrm{Hom}_S(\tau_1, \psi_S) + \dim \mathrm{Hom}_S(\tau_2, \psi_S) = 2.$$

Contradiction! □

Applications: Arthur packets

Idea (proposed by Wee Teck Gan at the 2022 AWS):
Use this correspondence to construct Arthur packets on G_2.

Applications: Arthur packets

Idea (proposed by Wee Teck Gan at the 2022 AWS):
Use this correspondence to construct Arthur packets on G_2.

Arthur parameters for G_2

$$\psi : WD_F \times SL_2 \to G_2$$

How does the Arthur SL_2 map into G_2?

Applications: Arthur packets

Idea (proposed by Wee Teck Gan at the 2022 AWS):
Use this correspondence to construct Arthur packets on G_2.

Arthur parameters for G_2

$$\psi : WD_F \times SL_2 \to G_2$$

How does the Arthur SL_2 map into G_2?

For "long root" Arthur packets, ψ factors through

$$\mathrm{SL}_3 \rtimes \mathbb{Z}/2\mathbb{Z} \subset G_2,$$

which is the dual group for PU_3.

Applications: Arthur packets

Idea (proposed by Wee Teck Gan at the 2022 AWS):
Use this correspondence to construct Arthur packets on G_2.

Arthur parameters for G_2

$$\psi : WD_F \times SL_2 \to G_2$$

How does the Arthur SL_2 map into G_2?

For "long root" Arthur packets, ψ factors through

$$\mathrm{SL}_3 \rtimes \mathbb{Z}/2\mathbb{Z} \subset G_2,$$

which is the dual group for PU_3.

$\Rightarrow$ we can use the $G_2 \times \mathrm{PU}_3$ correspondence to construct these G_2 packets by lifting from PU_3!

Thanks!