Large Values of the Riemann Zeta Function on the Critical Line

en l'honneur d'Andrew Granville

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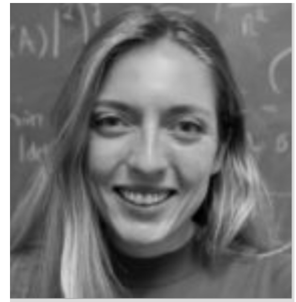
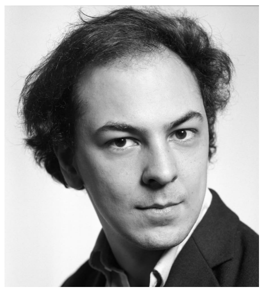
in collaboration with . . .



Emma Bailey



Maksym Radziwill

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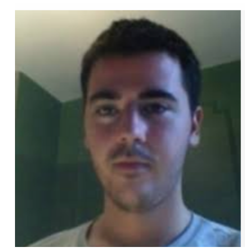
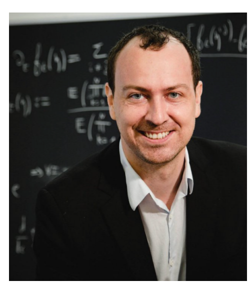


Frédéric Ouimet



Paul Bourgade

David Belius

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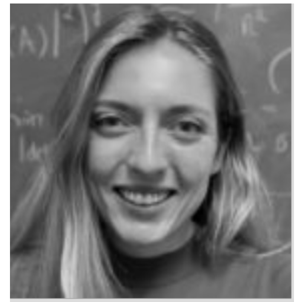

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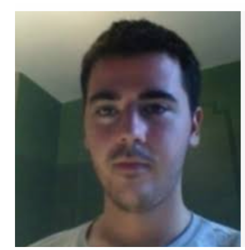

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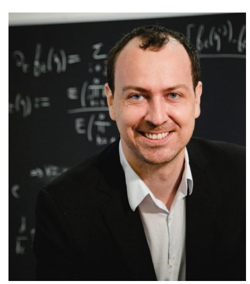

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Some useful facts on ζ

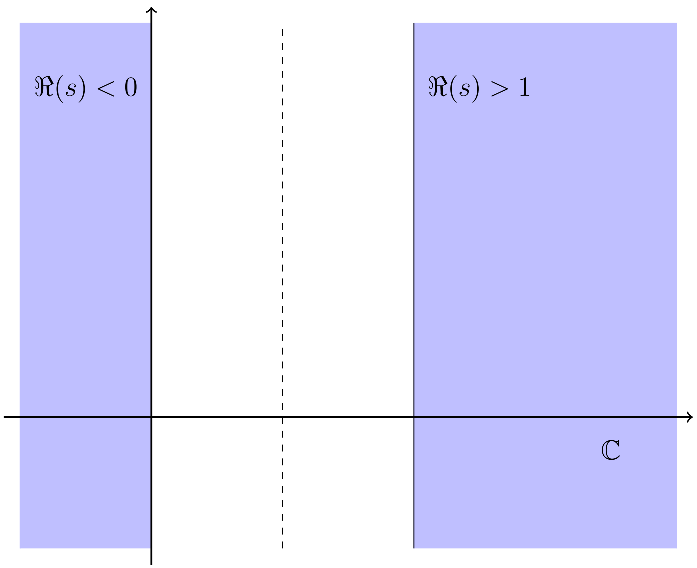
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} (1-p^{-s})^{-1}, \quad \mathrm{Re}\ s > 1$$

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi}{2}s\right) \Gamma(1-s)\zeta(1-s)$$

$$\zeta(1/2+\mathrm{i}t) = \sum_{n \le T} n^{-1/2-\mathrm{i}t} + \mathrm{O}(T^{-1/2}), \quad t \in [T, 2T]$$

Riemann von-Mangoldt:
$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + \mathrm{O}(\log T)$

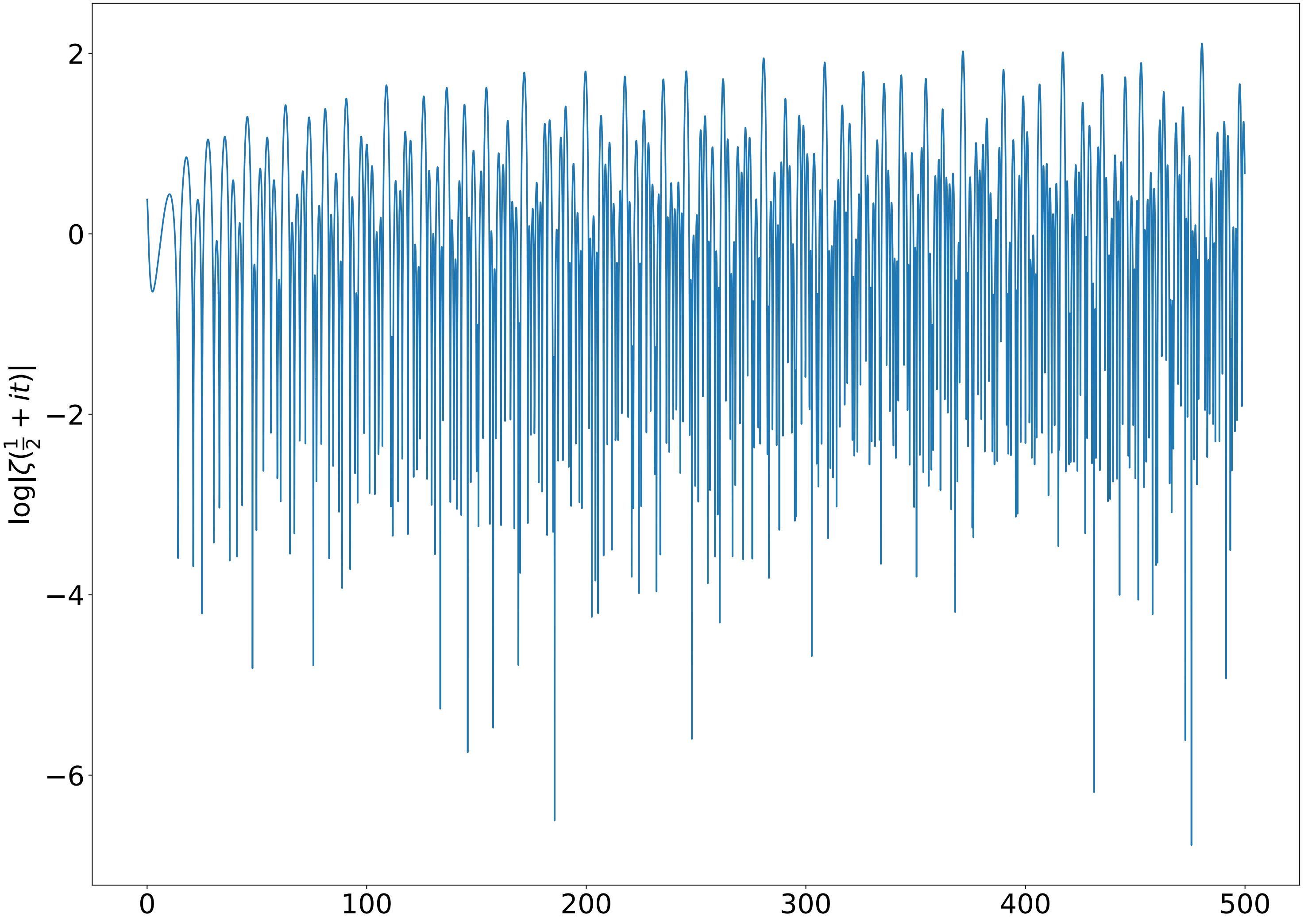
Gaps between zeros $\approx \frac{1}{\log T}$



Outline

1. Distribution of *typical values* on the critical line
2. Distribution of *large values* on the critical line
3. Large values in short intervals (if time permits)

1. Typical values on the critical line



Selberg's Central Limit Theorem

Theorem (Selberg's CLT)

Let τ be a uniform random variable on $[T, 2T]$, then

$$\frac{\log|\zeta(1/2+\mathrm{i}\tau)|}{\sqrt{\frac{1}{2}\log\log T}} \to \mathcal{N}(0,1).$$

What about values beyond $\sqrt{\log\log T}$?

$$\mathbf{P}(\log|\zeta(1/2+\mathrm{i}\tau)| > V) \overset{?}{\sim} \int_V^\infty \frac{e^{-z^2/\log\log T}}{\sqrt{\pi\log\log T}}\mathrm{d}z$$

Radziwiłł '11: Holds up to $V \ll (\log\log T)^{3/5-\varepsilon}$

Selberg's Central Limit Theorem

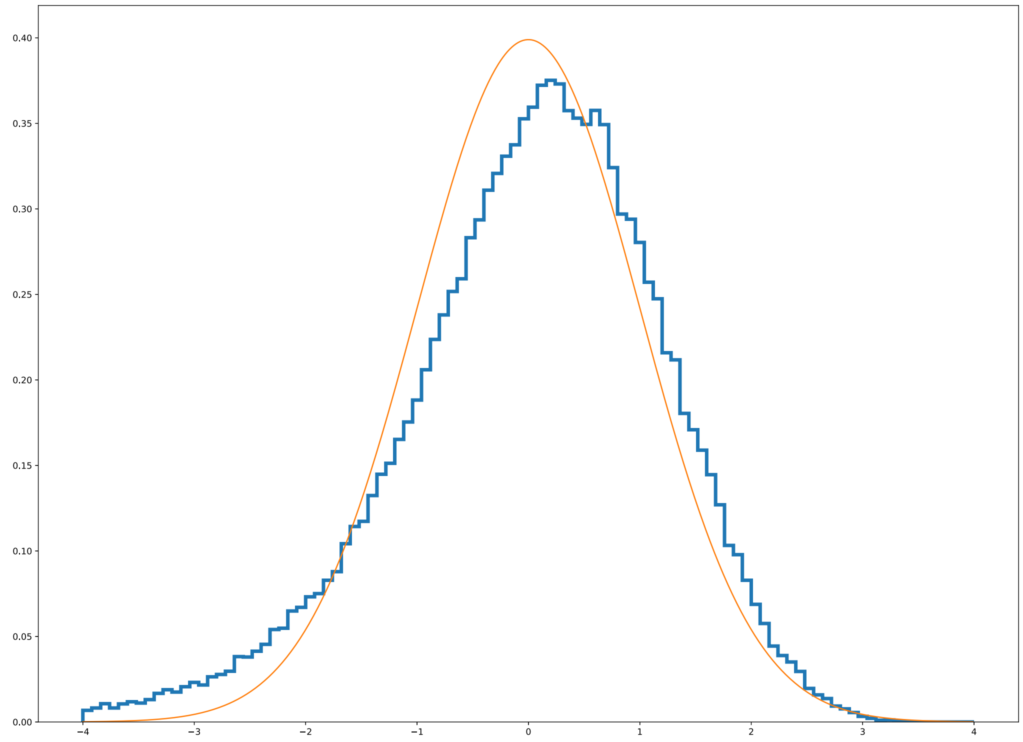
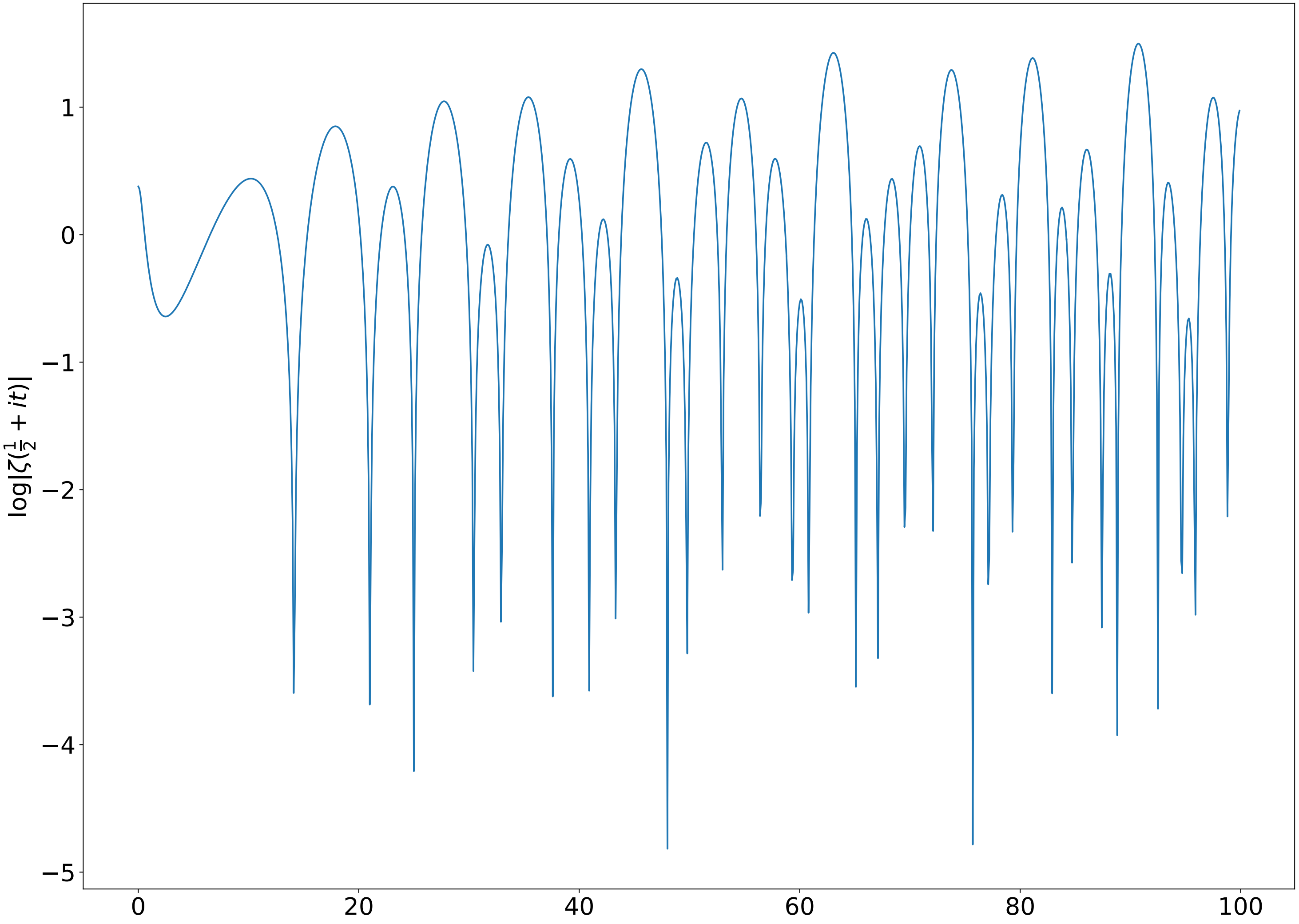
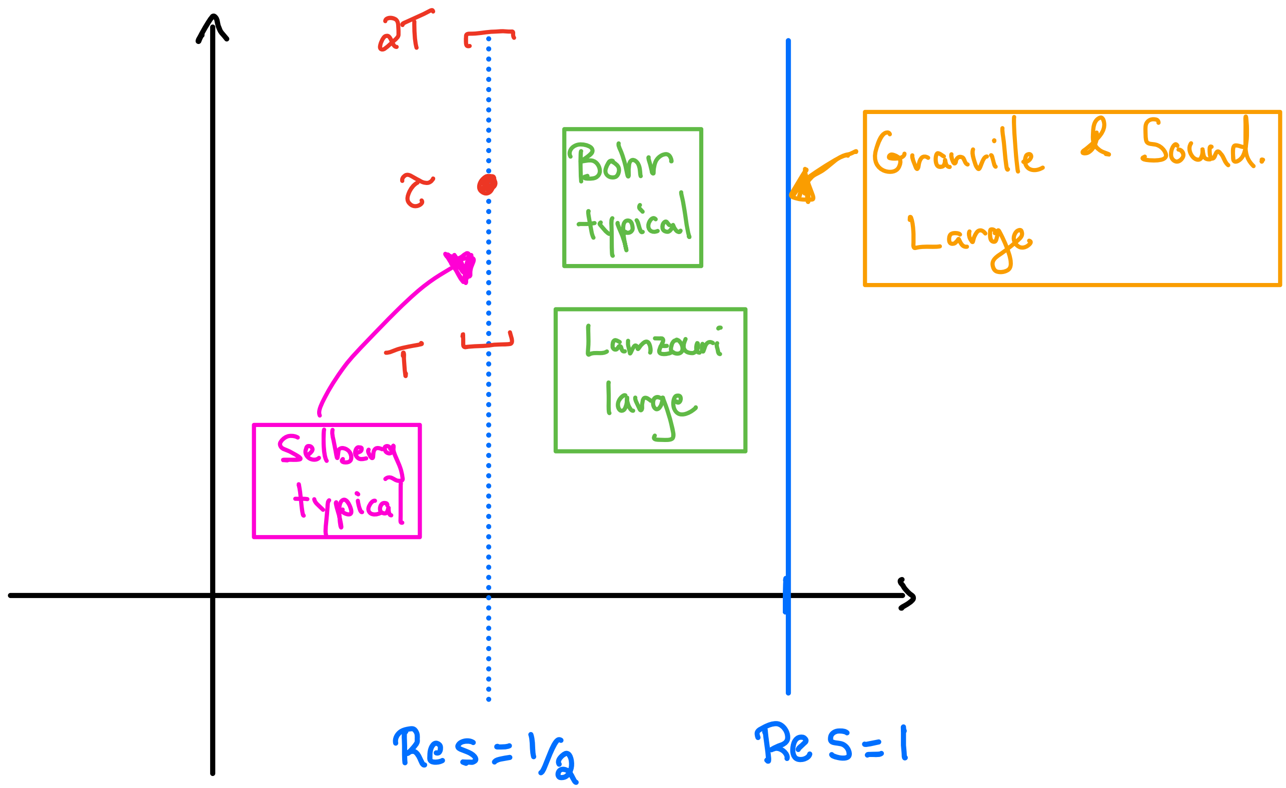


Figure: Distribution of 10000 values of $\frac{\log|\zeta(1/2+\tau)|}{\sqrt{1/2\log\log T}}$

2. Large values on the critical line



Large values in the critical strip



- Granville & Sound. '05: Distribution of $\log|\zeta(1+i\tau)|$ for V up to $\log_3 T$
- Lamzouri '11: Distribution of $\log|\zeta(\sigma+i\tau)|$ for V up to $\ll \frac{(\log T)^{1-\sigma}}{\log\log T}$ for $1/2 < \sigma < 1$

Large values on the critical line

Lindelöf hypothesis: $\mu(1/2) = 0$ or

$$M_k(T) = \frac{1}{T}\int_T^{2T} |\zeta(1/2+\mathrm{i}t)|^{2k}\mathrm{d}t = \mathbf{E}[|\zeta(1/2+\mathrm{i}\tau)|^{2k}] \ll T^{\varepsilon}$$

Moments Conjecture:

$$\mathbf{E}[|\zeta(1/2+\mathrm{i}\tau)|^{2k}] \sim C_k(\log T)^{k^2}.$$

for $C_k = a_k f_k$, f_k is predicted by RMT (Keating & Snaith '00).

Radziwiłł's Conjecture: If $V \sim k \log\log T$, $k > 0$,

$$\mathbf{P}(\log|\zeta(1/2+\mathrm{i}\tau)| > V) \sim C_k \int_V^{\infty} \frac{e^{-z^2/\log\log T}}{\sqrt{\pi \log\log T}}\mathrm{d}z.$$

Large values on the critical line

- Sound. '09 and Harper '12 showed on RH that

$$M_k(T) \ll (\log T)^{k^2+\varepsilon}, \quad k > 0$$

- Radizwiłł , Heap & Sound. '19 proved the above unconditionally for $0 < k \leq 2$.
- Lower bounds were proved by Radziwiłł & Sound. '13 and Heap & Sound. '20

$$\mathbb{E}\left[|\zeta(1/2+i\tau)|^{2k}\right] = \int_{\mathbb{R}} e^{2kV}\, \mathbb{P}(\log|\zeta(1/2+i\tau)| \in dV)$$

$$= 2k\int_0^\infty e^{2kV}\, \mathbb{P}(\log|\zeta(1/2+i\tau)| > V)\, dV + O(1)$$

2k-th moment $\longleftrightarrow$ $V \approx k \log\log T$

Large values on the critical line

Theorem (A-Bailey '22)

Let $V \sim \alpha \log\log T$ *with* $0 < \alpha < 2$. *We have*

$$\mathbf{P}(\log|\zeta(1/2 + \mathrm{i}\tau)| > V) \ll_\alpha \frac{1}{\sqrt{\log\log T}} e^{-V^2/\log\log T}.$$

In particular, this gives a new proof of the fractional moment bound of RHS '19.

$$\int_0^\infty e^{2kV} P(\log|\zeta(1/2+i\tau)| > V)\, dV \ll \int_0^{2\log\log T} e^{2kV} \frac{e^{-V^2/\log\log T}}{\sqrt{\log\log T}} + \text{Error}$$

$$\ll (\log T)^{k^2}$$

Sketch of Proof

Granville & Sound. on Re $s = 1$: compare ζ with a short (random) Euler product: $p \le \log T$.

Here we need $p \le T^{1/100}$.

$$S_k = \sum_{p \le \exp(e^k)} \frac{\mathrm{Re}(p^{-\mathrm{i}\tau})}{p^{1/2}}, \quad k \le \log\log T$$

- Think of $(S_k, k \ge 1)$ as a random walk:

$$S_{k+1} - S_k \approx \text{ IID } \mathcal{N}(0, 1/2)$$

Mertens

- For k close to $\log\log T$, S_k should be a good approximation for $\log|\zeta(1/2 + \mathrm{i}\tau)|$.

Sketch of Proof

Moments of S_k

$$\mathbf{E}\left[S_k^{2q}\right] = \underbrace{\frac{(2q)!}{2^q q!}(k/2)^q}_{\text{Gaussian moments}} + \mathrm{O}\left(\frac{\exp(2qe^k)}{T}\right)$$

- Fact: For large deviation $V \sim \alpha \log\log T$, need q of the order of $\log\log T$.
- Problem! For such q's can only take $k < \log_2 T - c\log_3 T$ or

$$p \le T^{1/(\log\log T)^c}.$$

How to deal with longer Dirichlet polynomials?

Sketch of Proof

Consider the event $H = \{\log|\zeta(1/2 + \mathrm{i}\tau)| > V\}$.
Define for $1 \leq \ell \leq \mathcal{L}$

$$T_\ell = T^{1/(\log_{\ell+1} T)^c} \qquad k_\ell = \log\log T_\ell = \log\log T - c \underbrace{\log_{\ell+2} T}$$

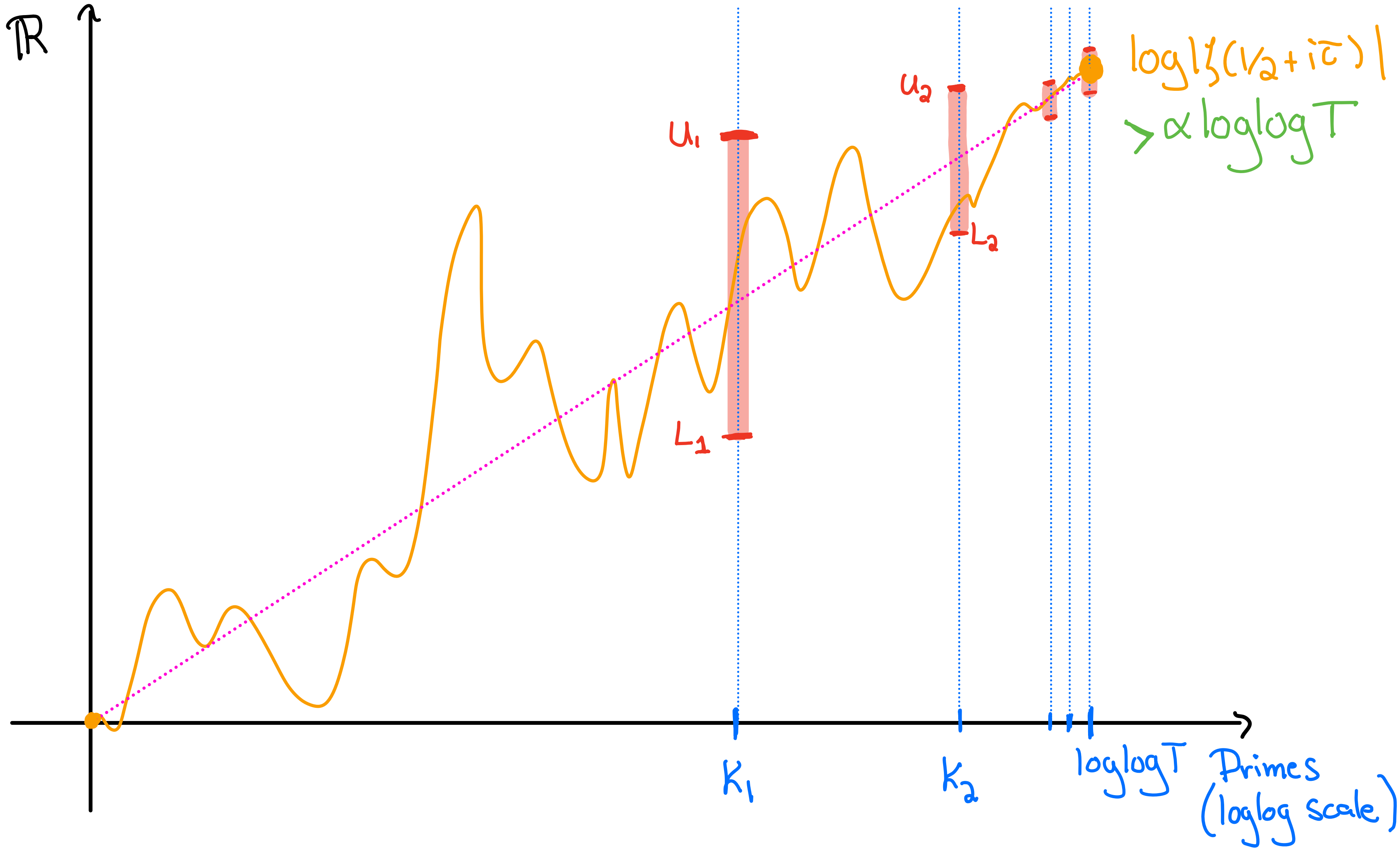
We also consider the good events

$$G_\ell = \{S_{k_j} \in [L_j, U_j] \; \forall j \leq \ell\}.$$

$$U_\ell = \alpha k_\ell + \mathcal{B} \underbrace{\log_{\ell+2} T} \qquad L_\ell = \alpha k_\ell - C \underbrace{\log_{\ell+2} T}$$

$$\mathbf{P}(H) = \mathbf{P}(H \cap G_1^c) + \sum_{1 \leq \ell \leq \mathcal{L}} \mathbf{P}(H \cap G_\ell \cap G_{\ell+1}^c) + \underbrace{\mathbf{P}(H \cap G_{\mathcal{L}})}_{\text{Dominant term}}$$

Sketch of Proof



Evaluating the dominant term

$$\mathbf{P}(H \cap G_{\mathcal{L}}) =$$

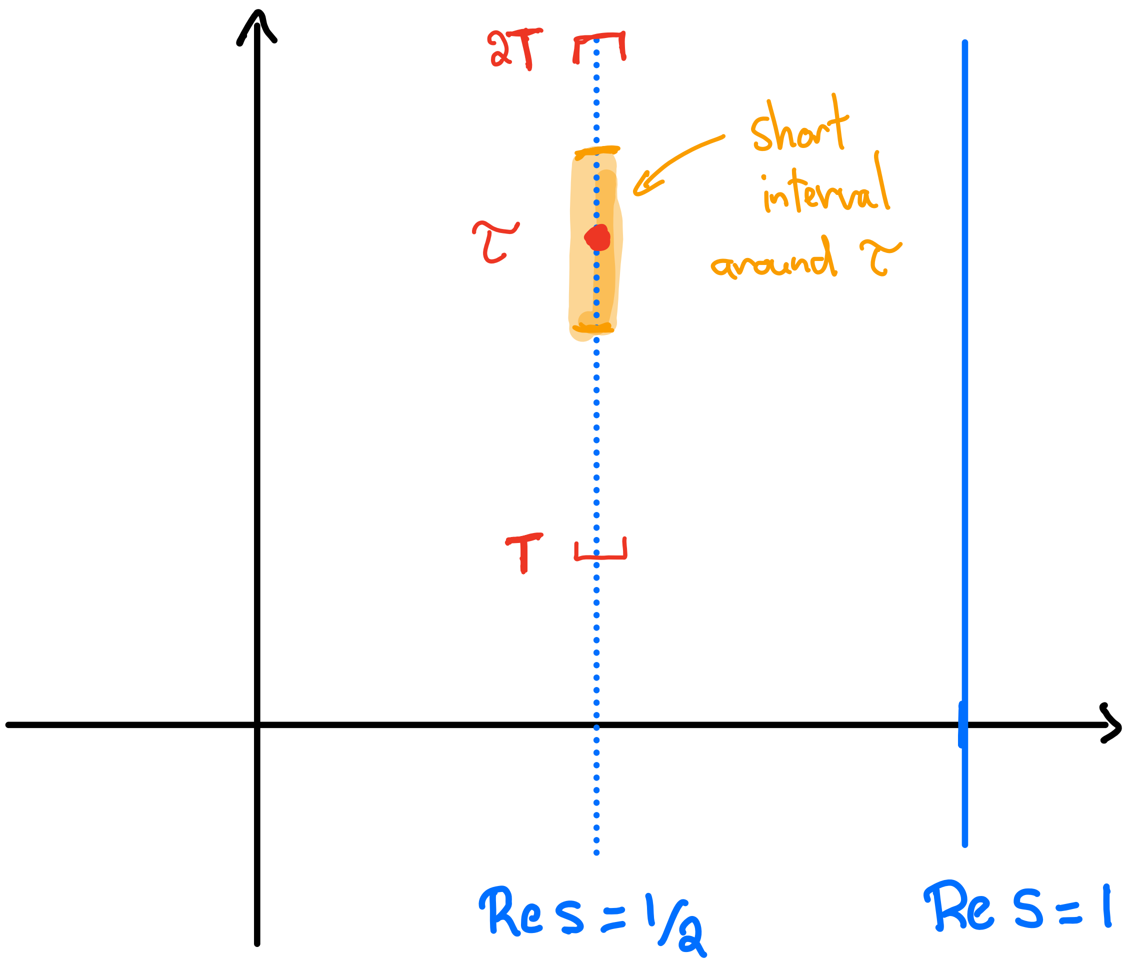
$$\sum_{v \in [L_{\mathcal{L}}, U_{\mathcal{L}}]} e^{-4(V-v)} \mathbf{E}\Big[|\zeta e^{-S_{k_{\mathcal{L}}}}|^4 \mathbf{1}(S_{k_{\mathcal{L}}} \in [v, v+1], S_{k_j} \in [L_j, U_j], j \leq \mathcal{L})\Big]$$

1. $\zeta e^{-S_{k_{\mathcal{L}}}} \approx \zeta \mathcal{M}_1 \dots \mathcal{M}_{\mathcal{L}}$
2. $\mathbf{1}(S_{k_{\mathcal{L}}} \in [v, v+1], S_{k_j} \in [L_j, U_j], j \leq \mathcal{L}) \approx Q$
   Dirichlet polynomial with primes $\leq T_{\mathcal{L}}$
3. $\mathbf{E}[|\zeta \mathcal{M}_1 \dots \mathcal{M}_{\mathcal{L}}|^4 |Q|^2] \approx \mathbf{E}[|\zeta \mathcal{M}_1 \dots \mathcal{M}_{\mathcal{L}}|^4] \cdot \mathbf{E}[|Q|^2]$ Twisted 4th moment
4. $\mathbf{P}(S_{k_{\mathcal{L}}} \in [v, v+1], S_{k_j} \in [L_j, U_j], j \leq \mathcal{L})$ can be evaluated

$$|S_{k_{\ell+1}} - S_{k_\ell}| \leq U_{\ell+1} - L_\ell \asymp \log_{\ell+2} T$$

Order of the variance!

3. Maximum in short intervals



Maximum in short intervals

Consider intervals of size $(\log T)^\theta$ on the critical line

T^θ would be great!

Corollary (A-Bailey '22)

Let $0 \le \theta < 3$. *We have for* $y = \mathrm{o}(\log\log T)$

$$\max_{h \le (\log T)^\theta} |\zeta(1/2 + \mathrm{i}(\tau + h))| > e^y \frac{(\log T)^{\sqrt{1+\theta}}}{(\log\log T)^{1/(4\sqrt{1+\theta})}}$$

with probability $\ll \underbrace{e^{-2\sqrt{1+\theta}y}}_{\text{Exponential}} \cdot \underbrace{e^{-y^2/\log\log T}}_{\text{Gaussian}}$.

Proof.

By the theorem and Riemann-von Mangoldt, the probability is

$$\ll (\log T)^{1+\theta} \cdot \frac{e^{-V^2/\log\log T}}{\sqrt{\log\log T}}$$

for $V = \sqrt{1+\theta}\log_2 T - \frac{1}{4\sqrt{1+\theta}}\log_3 T + y$. □

Maximum in short intervals

Fyodorov-Hiary-Keating Conjecture '12:

$$\max_{|h|\leq 1} |\zeta(1/2 + i(\tau + h))| = \frac{\log T}{(\log\log T)^{3/4}} \cdot e^{\mathcal{M}_T}$$

where $\mathcal{M}_T \to \mathcal{M}$ in distribution.

Theorem (A-Bourgade-Radziwiłł '20 '22+)

$$\max_{h\leq 1} |\zeta(1/2 + \mathrm{i}(\tau + h))| > e^{y} \frac{\log T}{(\log\log T)^{3/4}}$$

with probability $\ll y \cdot e^{-2y} \cdot e^{-y^2/\log\log T}$, $y = \mathrm{o}(\log\log T)$.

$$\max_{h\leq 1} |\zeta(1/2 + \mathrm{i}(\tau + h))| < e^{-y} \frac{\log T}{(\log\log T)^{3/4}}$$

with probability $\mathrm{o}(1)$ *as* $T \to \infty$ *and* $y \to \infty$.

Maximum in short intervals

Fyodorov-Hiary-Keating Conjecture '12:

$$\max_{|h|\leq 1} |\zeta(1/2 + i(\tau + h))| = \frac{\log T}{(\log\log T)^{3/4}} \cdot e^{\mathcal{M}_T}$$

where $\mathcal{M}_T \to \mathcal{M}$ in distribution.

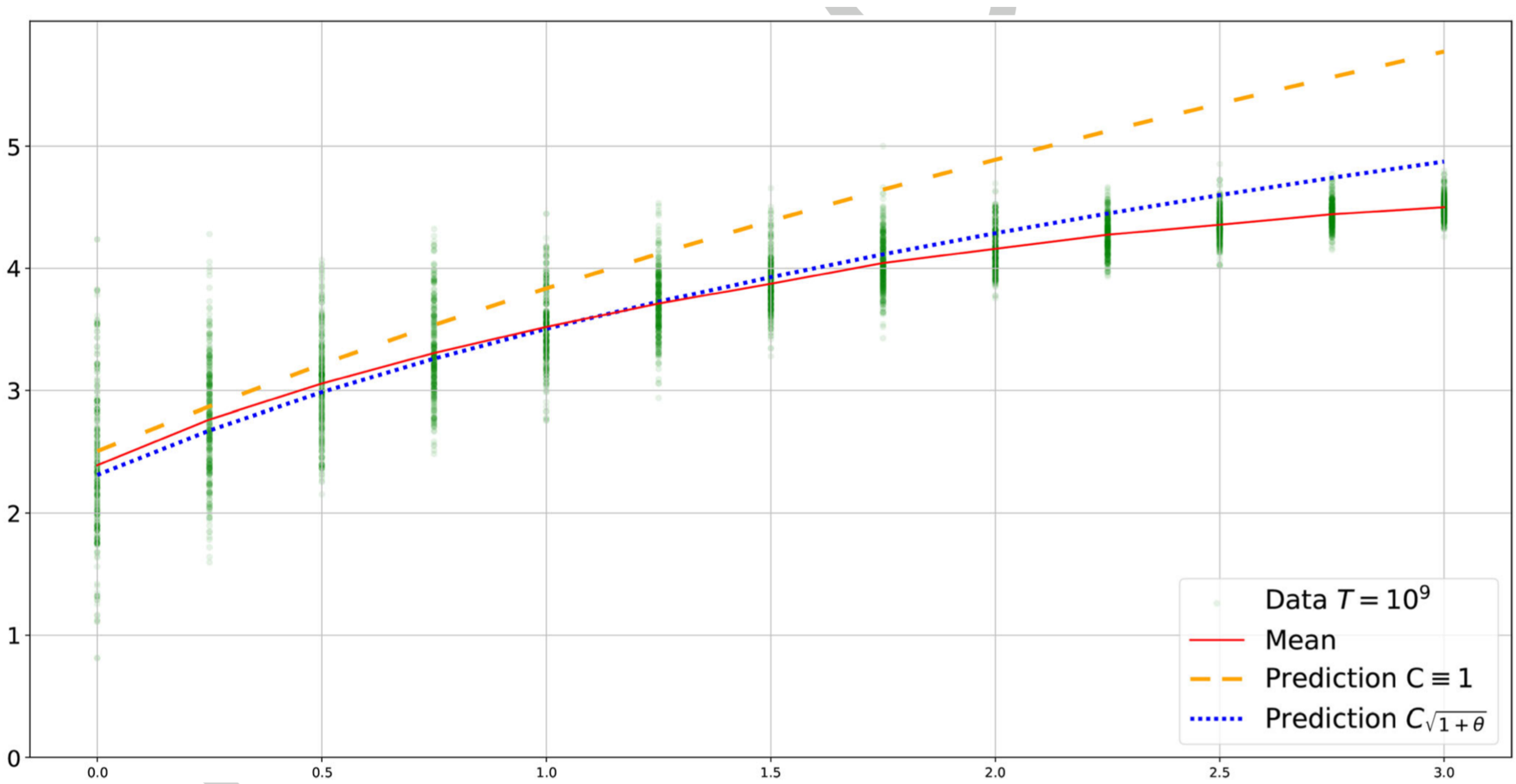
$$S_k(h) = \sum_{p \leq \exp(e^k)} \frac{\mathrm{Re}(p^{-\mathrm{i}(\tau+h)})}{p^{1/2}}, \quad k \leq \log\log T, \quad h \in [-1, 1]$$

Random walks are log-correlated.

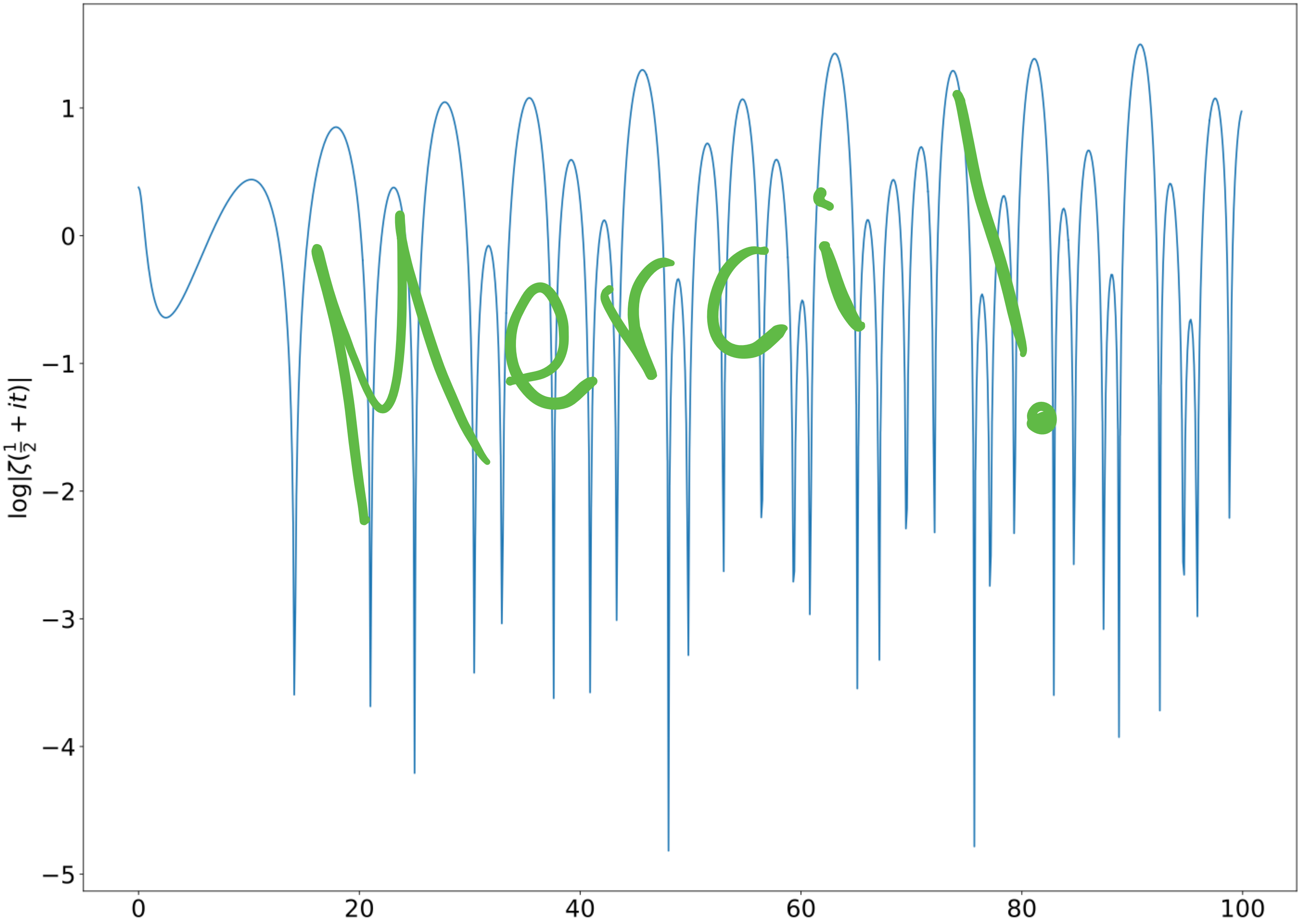
$$\mathbf{E}[S_k(h)S_k(h')] = \frac{1}{2} \sum_{p \leq \exp(e^k)} \frac{\cos(|h - h'| \log p)}{p} \sim \frac{1}{2} \log(|h-h'|^{-1} \wedge e^k)$$

Numerics!

Mean of $\max_{|h|\leq(\log T)^\theta} \log|\zeta(1/2+i(\tau+h))|$ as a function of θ



Arguin-A-Bailey-Hui-Rao '21



Merci.
$\log|\zeta(\frac{1}{2}+it)|$
1
0
−1
−2
−3
−4
−5
0
20
40
60
80
100