# Distributions,

# Differential Equations,

# and Zeros...

!

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∃ proofs

Some technical and historical background:
*Designed Pseudo-Laplacians*
E. Bombieri, P. Garrett
arXiv:2002.07929v1, 18 Feb 2020 (since 2013)
or, equivalently,
http://www.math.umn.edu/~garrett/m/v/
Bombieri-Garrett_current_version.pdf

+ "book" (∃ on-line)

$C_c^\infty(\Gamma\backslash\mathfrak{H}) = C_c^\infty(\Gamma\backslash G)^K$

## Simple case:

$\Gamma = SL_2(\mathbb{Z})$, invariant Laplacian $\Delta = y^2(\partial_x^2 + \partial_y^2)$ on $\mathfrak{H}$, descending to $\Gamma\backslash\mathfrak{H}$.

Let $\theta$ be a compactly-supported distribution on $\Gamma\backslash\mathfrak{H}$. Abbreviate $\lambda_s = s(s-1)$. Let

$$E_s(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\operatorname{Im}\gamma z)^s = \tfrac{1}{2}\sum_{\gcd(c,d)=1} \frac{y^s}{|cz+d|^{2s}}$$

**Theorem 1:** For $\mathrm{Re}(s) = \frac{1}{2}$, $(\Delta - \lambda_s)u = \theta$ has an $L^2$ solution $\Longrightarrow \theta(E_s) = 0$.

($H^{-\infty}$ ok: later)

True + proven

This is interesting because periods of Eisenstein series are *sometimes* zeta-functions ~~of~~ or $L$-functions: (Hecke-Maaß, *et al*): for $\theta$ the automorphic Dirac $\delta$ at $i \in \Gamma\backslash\mathfrak{H}$,

$$\theta(E_s) = \frac{\zeta_{\mathbb{Q}(i)}(s)}{\cancel{\zeta(2s)}}$$

on Re(s) = 1/2

edge

Logic ... ?!

More generally, for fundamental discriminant $d < 0$ with associated Heegner points $z_j$,

completed

$$\sum_j E_s(z_j) = \frac{\xi_{\mathbb{Q}(\sqrt{d})}(s)}{\xi_{\mathbb{Q}}(2s)}$$

For fundamental discriminant $d > 0$ with associated geodesic cycles $C_j$,

$$\sum_j \int_{C_j} E_s(h)\, dh = \frac{\xi_{\mathbb{Q}(\sqrt{d})}(s)}{\xi_{\mathbb{Q}}(2s)}$$

$\boxed{\textsf{Caution:}}$ Many periods $\theta(E_s)$ have off-line zeros.

For example, Epstein zetas $\delta_{z_o}(E_s) = E_s(z_o)$ have off-the-line zeros (Potter-Titchmarsh, Stark, *et al*).

1930's

1970's

$SL_2(\mathbb{C})$, $SU(2)$ periods ;

$\zeta_B(s) \to \zeta(2s)\,\zeta(2s-1)$ !?!

Landau

Granting spectral decomp'n of ?-fcns distr'ns?!

Trivial analogue: For perspective, consider $u'' - s^2 u = \delta$ on $\mathbb{R}$. By Fourier transform, for every $\mathrm{Re}(s) > 0$, there is an $L^2$ solution

not literal integral

$$\boxed{u(x) = \frac{e^{s|x|}}{-2s}}$$

$\hat{\delta} = 1$

Physics: physical corrob'n.

Math: scope—?!

Shewes' me…

But at $\mathrm{Re}(s) = 0$ the meromorphic continuation gives functions not in $L^2$.

In fact, by the theorem, via Fourier Inversion in place of spectral synthesis of automorphic forms, if there were an $L^2$ solution for some $\mathrm{Re}(s) = 0$, then $\delta(x \to e^{sx}) = 0$, so $1 = 0$, impossible.

appearing in spec. synthesis

Anyway, we did not expect to prove that $x \to e^{sx}$ had zeros.

haha

on [0,1]

## Continuing in the trivial context...

The *Sturm-Liouville* problem (reformulated)

not just 1 δ —

$$u'' - s^2 u \;=\; \overset{+}{-}\delta_1 + \delta_0 \qquad\qquad \text{(on } \mathbb{R}\text{)}$$

non-classical!

has an $L^2$ solution for infinitely-many eigenvalues $s^2 \;\leq\; 0$. The inhomogeneity supported at $\{0, 1\}$ reflects non-smoothness at the boundary of $[0, 1]$, described otherwise in classical discussions.

For $s \in i\mathbb{R}$ and a solution $u \in L^2(\mathbb{R})$, the theorem gives

$$(\delta_1 - \delta_0)(x \to e^{sx}) \;=\; 0$$

Thus,

$$e^s - e^0 \;=\; 0$$

which constrains $s$.

Double-check!

exemplify

case

**Remark:** Of course, explicit solutions

$$u(x) \;=\; \begin{cases} \sin(2\pi i n x) & (\text{for } 0 \le x \le 1) \\ 0 & (\text{otherwise}) \end{cases}$$

corroborate the conclusion. (The auto-duality of $\mathbb{R}$ makes this example nearly tautological.)

Sample

**Technicalities?** This trivial example $\mathbb{R}$ does illustrate certain technicalities:

A compactly supported distribution $\theta$ is tempered, so has a Fourier transform $\widehat{\theta}$. How to compute it? $\boxed{\widehat{\theta}(\xi) \;=\; \theta(x \;\to\; e^{-i\xi x})}$ is natural, but $x \to e^{i\xi x}$ is not Schwartz. It is smooth. Compactly supported distributions *are* (demonstrably) the dual $\mathcal{E}^*$ of $\mathcal{E} = C^\infty(\mathbb{R})$, so $\theta(x \to e^{-i\xi x})$ makes sense, ! … but *why* does it correctly compute the Fourier transform?

## Fourier inversion and $\theta \in \mathcal{E}^*$

on $\mathbb{R}$

For $u \in \mathscr{S}(\mathbb{R})$, by Fourier inversion

what sense ~~phrase~~? yeah!

$$u(x) = \int_{\mathbb{R}} e^{2\pi i \xi x}\, \widehat{u}(\xi)\, d\xi$$

In fact, with $\psi_\xi(x) = e^{2\pi i \xi x}$,

func coeffs

$$\mathscr{S} \ni u = \int_{\mathbb{R}} \psi_\xi\, \widehat{u}(\xi)\, d\xi \qquad (\mathcal{E}\text{-valued integral})$$

!

The integrand is not $\mathscr{S}$-valued. For $\theta \in \mathcal{E}^*$, by properties of $\boxed{\text{Gelfand-Pettis integrals}}$, 1930's

$\mathcal{E}$-valued

$$\theta(u) = \theta\Big(\int_{\mathbb{R}} \psi_\xi\, \widehat{u}(\xi)\, d\xi\Big)$$

$\mathcal{E}$-valued

$\mathcal{E}$-valued

!

$$= \int_{\mathbb{R}} \theta\big(\psi_\xi\, \widehat{u}(\xi)\big)\, d\xi \overset{\text{elem.}}{=} \int_{\mathbb{R}} \theta(\psi_\xi)\, \widehat{u}(\xi)\, d\xi$$

$\in \mathbb{C}$

By *uniqueness*, $\widehat{\theta}$ is a pointwise-valued function and $\widehat{\theta}(\xi) = \theta(x \to e^{-2\pi i \xi x})$.

Sketch

GL-2

## A little more generally:

(1/cA, not all periods of Eis — RH.)

$k$ a number field

$G = GL_2$ over $k$,
$K_v$ standard local maximal compact in
$G_v = GL_2(k_v)$, $K = \prod_{v \le \infty} K_v$.

Let $\Omega$ be among the $G_\infty$-invariant elements $(U\mathfrak{g})^G$ of the universal enveloping algebra $U\mathfrak{g}$ of the Lie algebra of $G_\infty$.

not just $\Delta$ ~ Casimir

Let $\lambda_{s,\omega}$ be the eigenvalue of $\Omega$ on the $s, \omega$ principal series of $G_\infty = \prod_{v|\infty} G_v$ → $E_{s,\omega}$

For unramified Hecke character $\omega$ of $k$, let $E_{s,\omega}$ be the (level-one) Eisenstein series.

$\boxed{\text{Let } \theta \text{ be a compactly supported distribution on } Z_{\mathbb{A}} \backslash G_{\mathbb{A}} / K.}$

or $\not\equiv 0$ @ $\infty$? $E_s$ mod $gK$.

not "local"

## Global Sobolev spaces: on $\mathbb{R}$

Converge how? where?

We need large spaces of (generalized) functions in which spectral expansions make sense and can be manipulated. Spectral expansion characterizations are convenient.

For example, $H^r(\mathbb{R})$ is the Hilbert-space completion of $C_c^\infty(\mathbb{R})$ with respect to the norm

Plancherel

$$|f|^2_{H^r} = \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \cdot (1+\xi^2)^r \, d\xi$$

$L^2$

Big $=$

$$H^{-\infty}(\mathbb{R}) = \bigcup_{r \in \mathbb{R}} H^r = \operatorname{colim}_r H^r$$

Sobolev's imbedding/inequality is (global! "easy")

$$H^{k+\frac{1}{2}+\varepsilon}(\mathbb{R}) \subset C^k(\mathbb{R}) \qquad (\text{for every } \varepsilon > 0)$$

Thus, $H^\infty = \bigcap_r H^r = \bigcap_k C^k = C^\infty. = \mathcal{E}$

easy

As a corollary, compactly supported distributions are in $H^{-\infty}$.

## Global automorphic Sobolev spaces:

In the simplest case of waveforms on $\Gamma\backslash\mathfrak{H}$ with $\Gamma = SL_2(\mathbb{Z})$, the spectral decomposition/synthesis assertion for $f \in L^2(\Gamma\backslash\mathfrak{H})$ is

$$f = \sum_{\text{cfm } F} \langle f, F\rangle \cdot F + \frac{\langle f, 1\rangle \cdot 1}{\langle 1, 1\rangle}$$

$$+\frac{1}{4\pi i}\int_{(\frac{1}{2})} \langle f, E_s\rangle \cdot E_s \, ds$$

$L^2$ $\quad \in L^2 \quad 1 \quad$ coefs $\quad L^2 \quad$ not $L^2 \quad$ cont. $\quad$ ?!

where $F$ runs over an orthonormal basis of cuspforms. The pairings are suggested by the $L^2$ pairing, but since $E_s \notin L^2(\Gamma\backslash\mathfrak{H})$, as $e^{i\xi x} \notin L^2(\mathbb{R})$, there are subtleties.

Tractible

Sobolev norms are

$$|f|^2_{H^r} = \sum_{\text{cfm } F} |\langle f, F\rangle|^2 \cdot (1+|\lambda_F|)^r + \frac{\langle f, 1\rangle \cdot 1}{\langle 1, 1\rangle}$$

$$+\frac{1}{4\pi i}\int_{(\frac{1}{2})} |\langle f, E_s\rangle| \cdot (1+|\lambda_s|)^r \, ds$$

$\exists$ Plancherel for $r=0$

... and $H^r = H^r(\Gamma\backslash\mathfrak{H})$ is the $H^r$-norm Hilbert space completion of $C_c^\infty(\Gamma\backslash\mathfrak{H})$.

Big $H^{-\infty} = \bigcup H^r = \mathrm{colim} H^r$

By design, every generalized function in $H^{-\infty}$ admits a spectral expansion of the same shape as for $L^2$. Luckily, $\mathcal{E}^* \subset H^{-\infty}$: by an automorphic version of Sobolev's lemma, $H^\infty \subset C^\infty(\Gamma\backslash\mathfrak{H}) = \mathcal{E}(\Gamma\backslash\mathfrak{H})$. Dualizing, $\mathcal{E}^* \subset H^{-\infty}$.

**Theorem 2:** For $\mathrm{Re}(s) = \frac{1}{2}$ and $\theta$ compactly supported, if $(\Omega - \lambda_{s,\omega})u = \theta$ has an $H^{-\infty}$ solution, then $\theta(E_{s,\omega}) = 0$.

Not cvgt ptwise!!!

Converge in $H^{-\infty}$ (?!?)

**Recall:** for quadratic $\ell/k$, the $GL_1(\ell)$ periods of $GL_2(k)$ Eisenstein series are

$$\int_{\mathbb{J}_k \ell^\times \backslash \mathbb{J}_\ell} E_{s,\omega}(h)\, dh \;\approx\; \frac{\Lambda_\ell(s, \omega \circ N_k^\ell)}{\Lambda_k(2s, \omega)}$$

$$\int_{\mathbb{J}_k \ell^\times \backslash \mathbb{J}_\ell} \chi(h) \cdot E_{s,\omega}(h)\, dh \approx \frac{\Lambda_\ell(s, \chi \cdot (\omega \circ N_k^\ell))}{\Lambda_k(2s, \omega)}$$

for Hecke character $\chi$ on $\mathbb{J}_\ell$ trivial on $\mathbb{J}_k$.

**But** not every period is a genuinely arithmetic object: generic Epstein zetas.

(e.g., see "book" __ )

do not sat. ~~[illegible]~~

RH

(stat.?! __ )

**Proof of theorem 1:** Write a spectral expansion of $\theta$, but only pay attention to the continuous-spectrum part:

contrary to typical

$$\theta \;=\; \ldots + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \widehat{\theta}(w)\cdot E_w\, dw$$

!? $\left( \widehat{\theta}(w) = \theta(E_w) \right)$ (!)

Since $\theta$ is compactly supported and $E_w$ is smooth, one can show that $\widehat{\theta}(w) = \theta(E_{1-w})$. Also,

$$u \;=\; \ldots + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \widehat{u}(w)\cdot E_w\, dw$$

and by properties of vector-valued integrals, the differentiation passes inside the integral:

$H^{-\infty}$ diff'n ! (more special than dist'n diff.?)

stable under "$\Delta$"

carry you outside your space

as R

$$
\begin{aligned}
&(\Delta - \lambda_s)u \\
&= \ldots + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \widehat{u}(w)\cdot(\Delta - \lambda_s)E_w\,dw \\
&= \ldots + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \widehat{u}(w)\cdot(\lambda_w - \lambda_s)E_w\,dw
\end{aligned}
$$

From $(\Delta - \lambda_s)u = \theta$, equating spectral coefficients,

← wt'd $L^2$ ✓

$$
\boxed{(\lambda_w - \lambda_s)\cdot\widehat{u}(w) = \widehat{\theta}(w) = \theta(E_w)}
$$

Careful

Since $\widehat{u}$ is locally $L^2$, $\theta(E_w)$ vanishes in a strong sense at $w = s$, as claimed.

After straightening out the complex conjugations... ///

± ✓ sol'ns of DE's

⟹ on-the-line 0's?

see preprint

## Faddeev-Pavlov/Lax-Phillips example:

FP (1967) and LP (1976) showed that (the Friedrichs extension of) $\Delta$ restricted to waveforms with constant term vanishing above height $a \geq 1$ has *purely discrete* spectrum.

In particular, a significant part of the orthogonal complement to cuspforms now decomposes *discretely*, *in addition to* being integrals of Eisenstein series!

Let $\theta$ be constant-term-evaluated-at-height-$a \gg 1$. By the theorem, for $\lambda_s < -\frac{1}{4}$, new $\lambda_s$-eigenfunctions $u$ can occur only when

$$0 \;=\; \theta E_s \;=\; \underbrace{a^s + \frac{\xi(2s-1)}{\xi(2s)} a^{1-s}}_{\text{const. term}}$$

Unfortunately, the on-the-line zeros of $\theta E_s$ refer to $\zeta(s)$ at the *edges* of the critical strip.

This *does* show that for $\lambda_s < -\frac{1}{4}$ the new/exotic eigenfunctions are certain *truncated Eisenstein series* $\wedge^a E_s$ with $\theta E_s = 0$ and $\mathrm{Re}(s) = \frac{1}{2}$.

Not *all* truncated Eisenstein series... c. 1980

The fact that this incarnation of $\Delta$ has *non-smooth* eigenfunctions seems to contradict *elliptic regularity*. In fact, this extension-of-a-restriction of $\Delta$ is *not* an elliptic differential operator.

This is abstractly similar to Sturm-Liouville problems...

## Hejhal (1981) and CdV (1981,82,83)

considered $(\Delta - \lambda_s)u = \delta_\omega^{\text{afc}}$ and similar, with $\omega = e^{2\pi i/3}$. From earlier computations (Fay 1978, et al), Hejhal observed that there is a *pseudo-cuspform* solution for $\text{Re}(s) = \frac{1}{2}$ if and only if $E_s(\omega) = 0$.

(A pseudo-cuspform has *eventually* vanishing constant term, and *eventually* is an eigenfunction of $\Delta$.)

CdV looked at Sobolev space aspects of this, to try to legitimately use Friedrichs extensions to convert $\boxed{(\Delta - \lambda_s)u = \delta}$ to a *homogeneous* equation. This resembles P. Dirac's and H. Bethe's work c. 1930, on "*singular potentials*:"

$$((\Delta - \delta \otimes \delta) - \lambda_s)u = 0$$

many?

?! 100% of 0's on-the-line ?!

## Attempting to construct solutions:

The FP/LP and Hejhal/CdV examples are inspirational, and/but we hope for more. Our project has clarified CdV's 1982-3 further speculations a bit...

For *negative* fundamental discriminants (we proved) at most 94% of the on-line zeros of $\zeta(s)$ enter as discrete spectrum $s(s-1)$. Without assuming things in violent contrast to current belief systems, probably none. Also, *construction* of PDE solutions by *physical* means is unclear.

0%

physical

For *positive* discriminants, there is more hope to construct PDE solutions physically, since the Hecke-Maaß functionals involve integration over codimension-one cycles...

$GL_1$ (qu.-extn)
periods on $GL_2$-

**A too-simple attempt:** Take $k = \mathbb{Q}(\sqrt{d})$ with $d > 0$ and narrow class number one. Imbed $\mathbb{Q}(\sqrt{d}) \to M_2(\mathbb{Q})$ by a nice rational representation. Let $\Gamma = SL_2(\mathbb{Z})$, and let $U \subset SL_2(\mathbb{Z})$ be the image of units in $\mathfrak{o}$. Let $H$ be the real Lie group (a ~~circle~~ $\mathbb{R}^\times$) whose rational points are the image of norm-one elements of $\mathbb{Q}(\sqrt{d})$.

$H/U \simeq \odot$

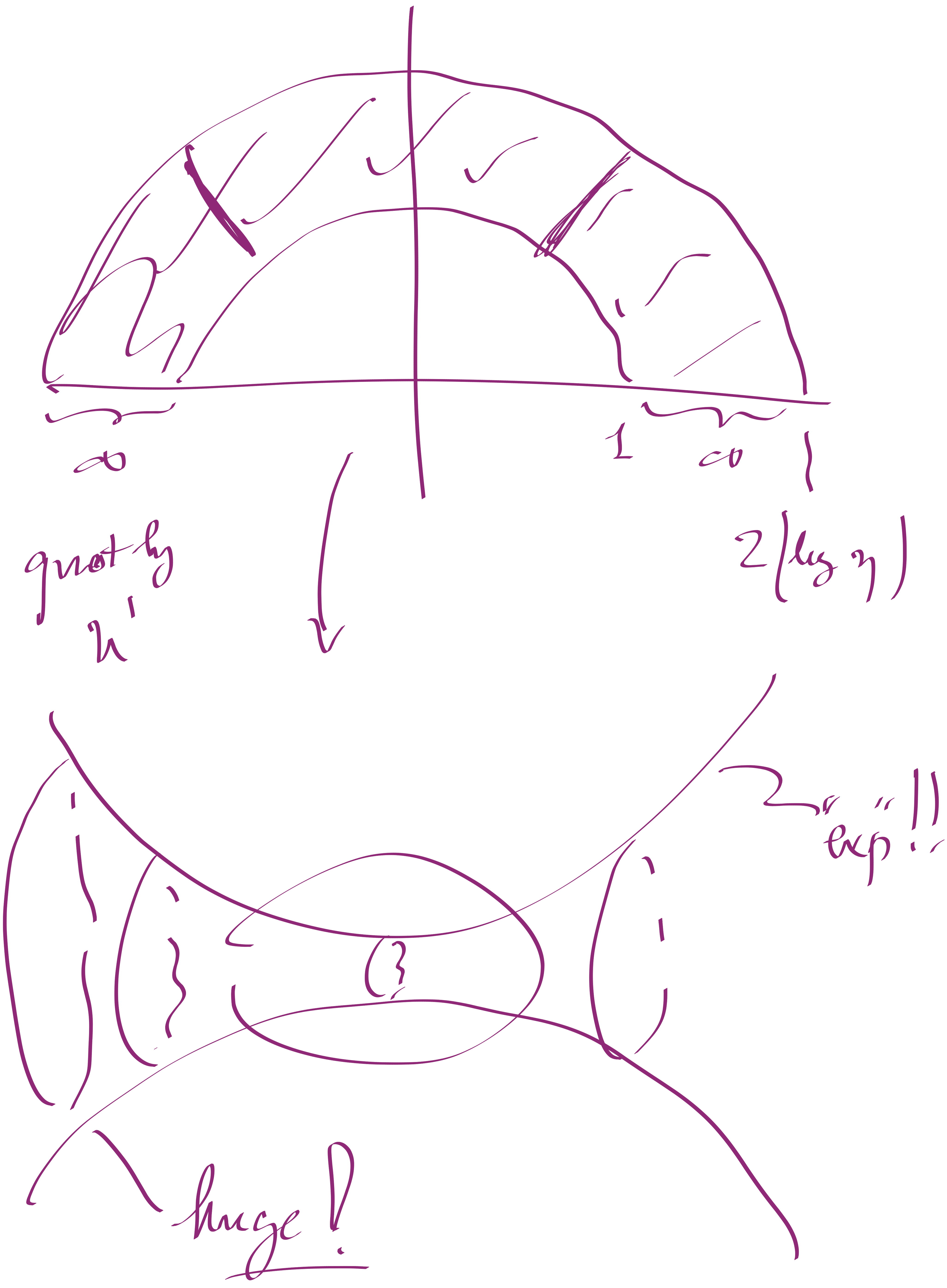
The quotient $U\backslash\mathfrak{H}$, a cylinder, naturally maps to the modular curve $\Gamma\backslash\mathfrak{H}$. The subgroup $U$ is conjugate in $SL_2(\mathbb{R})$ to the subgroup

$$U' = \left\{\begin{pmatrix} \eta^n & 0 \\ 0 & \eta^{-n} \end{pmatrix} : n \in \mathbb{Z}\right\}$$

for suitable $\eta \in \mathfrak{o}^\times$. $U'$ has a convenient fundamental domain

$$F = \{z \in \mathfrak{H} : 1 \leq |z| < 2|\log \eta|\}$$

Pictures:



check for low hanging fruit

Compatibly with the choice of fundamental domain $F$ for $U'$, in polar coordinates on $\mathfrak{H}$

$$\Delta^{\mathbb{R}^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2}{\partial \theta^2}$$

$$\Delta = \Delta^{\mathfrak{H}} = \sin^2\theta \cdot \left( r^2 \frac{\partial^2}{\partial r^2} + r \cdot \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2} \right)$$

Separate variables: take $u(r, \theta) = A(r) \cdot B(\theta)$, and require $\sin^2\theta \cdot B'' = \mu \cdot B$ for $\mu < 0$. The eigenvalue equation $\Delta u = \lambda \cdot u$ becomes

naive

$$r^2 \cdot A'' + r \cdot A' + (\mu - \lambda) \cdot A = 0$$

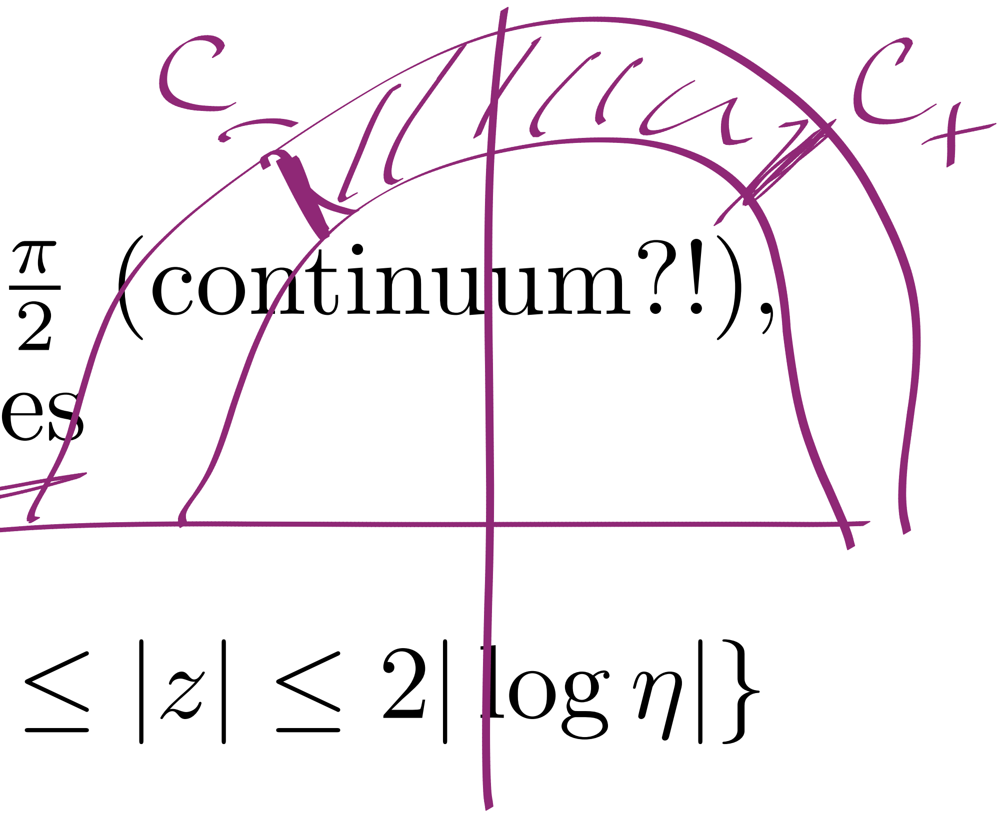
This Euler-type equation has solutions $r^\alpha$ for $\alpha(\alpha - 1) + \alpha + (\mu - \lambda) = 0$. The simplest sequel takes $\alpha = 0$, so $\lambda = \mu$.

to mind

We want a compactly-supported function $u$ on $\overline{F}$ that is radially invariant and satisfies

$$\sin^2\theta \cdot u''(\theta) = \lambda \cdot u(\theta) + C^+ + C^-$$

... where, for fixed $0 < a < \frac{\pi}{2}$ (continuum?!), $C^{\pm}$ are (integrals over) cycles




$$C^{\pm} = \{z : \arg z = \frac{\pi}{2} \pm a,\ 1 \leq |z| \leq 2|\log \eta|\}$$

We want $B(\frac{\pi}{2} \pm a) = 0$, and *symmetry* of $u$ under $\theta \to \frac{\pi}{2} - \theta$.

That is, the values $\mu = \lambda < 0$ are such that an *even* solution $B = B_\lambda$ of $\sin^2\theta \cdot B'' = \lambda \cdot B$ has zeros at $\theta = \frac{\pi}{2} \pm a$.

Being a Sturm-Liouville problem, there are infinitely-many such $\lambda$ (by alternation of roots), and an asymptotic (Weyl's Law).

For such $\lambda$,

$$u(r,\theta) = \begin{cases} B_\lambda(\theta) & (\text{for } \frac{\pi}{2} - a \leq \theta \leq \frac{\pi}{2} + a) \\ 0 & (\text{otherwise}) \end{cases}$$

*Winding-up/automorphizing* this compactly-supported $u$, and changing coordinates, gives a function on $\Gamma\backslash\mathfrak{H}$ (still denoted $u$) such that (up to a multiplicative constant)

$$(\Delta - \lambda)u \;=\; C^+ + C^-$$

By the theorem,

$$(C^+ + C^-)(E_s) \;=\; 0$$

$C^\pm E_s$ are Euler products, and differ from $\xi_k(s)/\xi(2s)$ only at the archimedean factor.

Good so far.

ominous

(can complete comp'n)

However, the archimedean factors are *perturbed* enough so that their sum can account for the forced zeros of the altered version(s) of $\xi_k(s)$.

Bessel / analogues of Sturm–L.

The continuum of choices of $a$ should have been ominous, too.

mimics λ's

Also, images of geodesics are not reliably the same things as subgroup orbits...

Done! Thanks!