Monotone Chains of Fourier Coefficients of Hecke Cusp Forms

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Arrangements Problem

$f : \mathbb{N} \to \mathbb{R}$ multiplicative (i.e., $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$)

Problem (Infinitely Many Solutions)

If $a_1, \ldots, a_k \geq 0$ are distinct integers then the set

$$\{n \in \mathbb{N} : f(n + a_1) < f(n + a_2) < \cdots < f(n + a_k)\} \tag{1}$$

is unbounded.

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Problem (Sharp Density)

The set (1) has natural density $1/k!$, i.e.,

$$\frac{1}{X}|\{n \leq X : f(n + a_1) < \cdots < f(n + a_k)\}| = \frac{1}{k!} + o_{X\to\infty}(1).$$

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- Distributions of $f(n + a_i)$, $f(n + a_j)$ are independent for "typical" multiplicative function; all arrangements should be equally likely
- For f unbounded (e.g., divisor function), $f(n + a_i) = f(n + a_j)$ is rare

Examples:

Some information can be gleaned if $k = 2$:

- for $f(n) = \sum_{d|n} 1$, we have $f(n) \geq 2$, with equality iff n is prime; then we have $f(p) < f(p-1)$ and $f(p) < f(p+1)$ i.o.
- **Erdős (1940's):** $f(n) < f(n+1)$ (resp. $f(n) > f(n+1)$) for all n iff $f(n) = n^\alpha$ with $\alpha > 0$ (resp. $\alpha < 0$)
- **Matomäki-Radziwiłł (2015):** If $a \neq 0$ then $f(n) < 0 < f(n+a)$ occurs for a positive proportion of n, provide $f(n) < 0$ has a solution

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For $k \geq 3$ this is already hard when f takes only positive values:

Conjecture (Sarkőzy, '00)

If $f : \mathbb{N} \to \mathbb{N}$ and f is not monotone then both

$$f(n) < \min\{f(n-1), f(n+1)\} \text{ and } f(n) > \max\{f(n-1), f(n+1)\}$$

occur i.o.

Fourier Coefficients of Cusp Forms

Focus on f arising from Fourier coefficients of arithmetically normalized Hecke cusp form ϕ (non-CM with trivial nebentypus):
$\phi(z) = \sum_{n\geq 1} f(n)e^{2\pi inz}$
For concreteness, take $\phi = \Delta$, where, writing $q = e^{2\pi iz}$,

$$\Delta(z) = q\prod_{m\geq 1}(1-q^m)^{24} = \sum_{n\geq 1}\tau(n)q^n,$$

so $f(n) = \tau(n)$ is the Ramanujan τ-function
Important Properties:

- $\tau(n) \in \mathbb{Z}$, multiplicative
- $|\tau(p)| \leq 2p^{11/2}$ (Deligne)
- $\{\tau(p)\}_p$ satisfies a Sato-Tate law: if $[a,b] \subseteq [-2,2]$,

$$\left|\left\{p \leq X : a \leq \frac{\tau(p)}{p^{11/2}} \leq b\right\}\right| = \pi(X)\left(\frac{2}{\pi}\int_a^b \sqrt{4-u^2}\,du + o_{X\to\infty}(1)\right)$$

Admissibility and Vanishing of τ

Let $\mathcal{N}_\tau := \{n \in \mathbb{N} : \tau(n) \neq 0\}$.
Lehmer's Conjecture: $\mathcal{N}_\tau = \mathbb{N}$
Serre: $\mathcal{N}_\tau$ has positive natural density
Definition: Let $k \geq 1$. A k-tuple $\boldsymbol{a} = (a_1, \ldots, a_k)$ is *admissible* if the a_j are distinct non-negative integers, such that for each $p \notin \mathcal{N}_\tau$ the set

$$\{m \pmod{p} : m \not\equiv a_j \pmod{p}\ \forall\ 1 \leq j \leq k\} \neq \emptyset.$$

Proposition

Let $k \geq 1$. If $\boldsymbol{a}$ is admissible then $\{n \in \mathbb{N} : n + a_j \in \mathcal{N}_\tau\ \forall\ 1 \leq j \leq k\}$ has positive density.

Given $\boldsymbol{a}$ admissible, by *relative density* of $S \subseteq \mathbb{N}$ we mean the limit

$$\lim_{X \to \infty} \frac{|S \cap \{n \leq X : n + a_j \in \mathcal{N}_\tau\ \forall j\}|}{|\{n \leq X : n + a_j \in \mathcal{N}_\tau\}|} \text{ (if it exists)}$$

Arrangement Problem with τ - $k = 2, 3$

Theorem (Klurman-M., '20+)

If (a_1, a_2) is admissible then the set

$$\{n \in \mathbb{N} : n + a_1, n + a_2 \in \mathcal{N}_\tau, \tau(n + a_1) < \tau(n + a_2)\}$$

has relative upper density $\geq 1/2$.

Theorem (Klurman-M., '20+)

Let $\boldsymbol{a} = (a_1, a_2, a_3)$ be admissible. Then the set

$$\{n \in \mathbb{N} : n + a_1, n + a_2, n + a_3 \in \mathcal{N}_\tau, \tau(n + a_1) < \tau(n + a_2) < \tau(n + a_3)\}$$

has relative upper density $\geq 1/6$.

The case $k = 3$ is completely new!

Conditional Result - $k > 3$

In general, we cannot say anything for $k > 3$, unless we assume an additional conjecture about correlations of *bounded* multiplicative functions:

Theorem (Klurman-M., '20+)

Assume Elliott's conjecture holds. Let $k \geq 2$ *and let* $(a_1, a_2, \ldots, a_k)$ *be admissible. Then*

$$\{n \in \mathbb{N} : \tau(n + a_1) < \cdots < \tau(n + a_k)\}$$

has relative natural density $1/k!$.

We discuss Elliott's conjecture shortly.

Proof Ideas: First Observations

For $n \in \mathcal{N}_\tau$ write $\tau(n) = |\tau(n)|\sigma(n)$, where $\sigma(n) := \text{sign}(\tau(n))$
Suppose $\tau(n+a_1) < \cdots < \tau(n+a_r) < 0 < \cdots < \tau(n+a_k)$, or let $r = 0$.
Then:

$$|\tau(n+a_i)| > |\tau(n+a_j)|, \sigma(n+a_i) = \sigma(n+a_j) = -1 \text{ for } 1 \leq i < j \leq r$$

$$|\tau(n+a_i)| < |\tau(n+a_j)|, \sigma(n+a_i) = \sigma(n+a_j) = +1 \text{ for } r+1 \leq i < j \leq k$$

<u>Questions to address:</u>

- How often do inequalities $|\tau(n+a_i)| > |\tau(n+a_{i+1})|$ occur for $1 \leq i \leq r-1$ (and same question in reverse for $r+1 \leq i < k$)?
- How often is $(\sigma(n+a_1), \ldots, \sigma(n+a_k)) = \boldsymbol{\epsilon}$, for $\boldsymbol{\epsilon} \in \{-1,+1\}^k$ with $\epsilon_j = -1$ for $1 \leq j \leq r$, $\epsilon_j = +1$ otherwise?
- How often do these conditions occur *simultaneously*?

Arrangement Problem with $|\tau|$

Theorem (Bilu-Deshouillers-Gun-Luca, '17)

Let $k \geq 1$. If $\boldsymbol{a}$ is admissible then

$$|\{n \leq X : 0 < |\tau(n+a_1)| < \cdots < |\tau(n+a_k)|\}| \gg_k X/(\log X)^k;$$

in particular, $|\tau(n+a_1)| < \cdots < |\tau(n+a_k)|$ i.o.

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$$\frac{|\{n \leq X : 0 < |\tau(n+a_1)| < \cdots < |\tau(n+a_k)|\}|}{|\{n \leq X : n+a_j \in \mathcal{N}_\tau \ \forall j\}|} = \frac{1}{k!} + o_{X\to\infty}(1).$$

Idea: Apply Erdős-Kac theorem to study random vector $(\log|\tau(\boldsymbol{n}+a_1)|, \ldots, \log|\tau(\boldsymbol{n}+a_k)|)$ for $\boldsymbol{n} \in [1,x]$ randomly chosen with $\boldsymbol{n}+a_j \in \mathcal{N}_\tau$, being careful with very small values of $|\tau(p)|$

Proof Ideas: Patterns of sign$(n + a_j)$

- Sato-Tate $\Rightarrow \sigma(n + a_j) = \pm 1$ with equal probability $1/2$
- If signs are independent, $(\sigma(n + a_1), \dots, \sigma(n + a_k)) = \epsilon$ should occur with probability $1/2^k$ for each $\epsilon \in \{-1, +1\}^k$

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- Since

$$1_{\sigma(n+a_j)=\epsilon_j} = \frac{1}{2}(1 + \epsilon_j \sigma(n + a_j)),$$

we can control sign patterns via correlations:

$$|\{n \leq X : \sigma(n+a_j) = \epsilon_j \ \forall 1 \leq j \leq k\}| = \sum_{n \leq X} \prod_{1 \leq j \leq k} \frac{1}{2}(1+\epsilon_j \sigma(n+a_j))$$

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$$= 2^{-k} \sum_{S \subseteq \{1,\ldots,k\}} \left(\prod_{j \in S} \epsilon_j \right) \sum_{n \leq X} \prod_{j \in S} \sigma(n + a_j).$$

<u>Question:</u> Are the sums $o(X)$ for all $S \neq \emptyset$?

Correlations of Multiplicative Functions

Question: For which $f : \mathbb{N} \to \mathbb{U}$ multiplicative is it the case that

$$\sum_{n \leq X} f(n)\overline{f}(n+a) \neq o(X)? \qquad (2)$$

Example 1: $f(n)$ is a Dirichlet character χ modulo a

Example 2: $f(n)$ is smooth and slowly-varying, e.g., $f(n) = n^{it}$, $t \in \mathbb{R}$

Heuristic: (2) holds iff f "behaves like" some $\chi(n)n^{it}$.

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Conjecture (Elliott's Conjecture)

Let X be large. Assume that for each fixed Dirichlet character χ we have

$$\min_{|t| \le X} \sum_{p \le X} \frac{1 - Re(f(p)\overline{\chi}(p)p^{-it})}{p} \to \infty \text{ as } X \to \infty.$$

Then for any distinct non-negative integers $a_1, \ldots, a_k$,

$$\sum_{n \le X} f(n+a_1) \cdots f(n+a_k) = o(X).$$

Partial Results Towards Elliott

By changing how we count, we have partial results for $k = 2, 3$:

Theorem (Tao, '15)

If $a \geq 1$ and f satisfies the condition in Elliott's conjecture then

$$\sum_{n \leq X} f(n)\overline{f}(n+a)/n = o(\log X).$$

Theorem (Tao-Teräväinen, '17)

If

$$\sum_{p \leq X} (1 - Re(f(p)^3\overline{\chi(p)}))/p \gg \log\log X$$

for all fixed Dirichlet characters χ then for any a_1, a_2 distinct positive integers,

$$\sum_{n \leq X} f(n)f(n+a_1)f(n+a_2)/n = o(\log X).$$

Proof Ideas: Handling Correlations of $\sigma(n)$

In the conditional and unconditional results, need to establish that sums

$$\sum_{p \le X} \frac{1 - \mathrm{Re}(\sigma(p)\overline{\chi}(p)p^{-it})}{p}$$

are growing with X (uniformly in $|t| \le X$).
Since σ is real-valued, it (roughly-speaking) suffices to consider $t = 0$ and χ real-valued.
Question: How does $\sigma(p) = \mathrm{sign}(\tau(p))$ behave in arithmetic progressions?

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Since σ is real-valued, it (roughly-speaking) suffices to consider $t = 0$ and χ real-valued.
Question: How does $\sigma(p) = \mathrm{sign}(\tau(p))$ behave in arithmetic progressions?
Answer: Using the breakthrough work of Newton-Thorne on automorphy of $L(s, sym^n\Delta)$, we establish quantitative Sato-Tate in arithmetic progressions, i.e., asymptotic for

$$\{p \le X : p \equiv a \pmod{q}, a \le \tau(p)p^{-11/2} \le b\},$$

for $q \le (\log\log X)^A$, $[a, b] \subseteq [-2, 2]$ (possibly tending to 0 with X).

Thanks for listening!

Proof Ideas: Distribution of $|\tau(n+a_j)|$s

For $n + a_j \in \mathcal{N}_\tau$ for all j,

$$|\tau(n+a_1)| < \cdots < |\tau(n+a_k)| \Rightarrow \log|\tau(n+a_1)| < \cdots < \log|\tau(n+a_k)|$$

$g_\tau(n) := \log|\tau(n)n^{-11/2}|$ is *additive*, i.e., $g_\tau(mn) = g_\tau(m) + g_\tau(n)$, for $(m,n) = 1$; have lots of tools available!
By a covering argument, it is enough to consider

$$\{n \leq X : n + a_j \in \mathcal{N} \ \forall j, g_\tau(n+a_j) \in I_j\} : I_j \subseteq \mathbb{R} \text{ intervals}$$

Rough Heuristic: Provided $|g_\tau(p)|$ is not "typically" too large on the primes, then g satisfies the Erdős-Kac theorem, i.e.,

$$\frac{1}{X}|\{n \leq X : \tilde{g}(n) \in I\}| = \frac{1}{\sqrt{2\pi}} \int_I e^{-u^2/2} du + o_{X\to\infty}(1),$$

where $\tilde{g}(n)$ is a centred and normalized version of $g(n)$.

Sieving with Sato-Tate

Problem: Very small values of $|\tau(p)|p^{-11/2}$ may occur...
Idea: Say $\xi(X) \to 0$ with X. We want to control

$$|\{p \leq X : 0 < |\tau(p)|p^{-11/2} < \xi(X)\}|.$$

Theorem (Thorner, '20+): Recent breakthrough of Newton-Thorne on automorphy for L-functions of all $Sym^n\Delta$ implies quantitative Sato-Tate!
Corollary: For all but $o(X)$ integers $n \leq X$,

$$\log|\tau(n)n^{-11/2}| \sim \log|\tilde{\tau}_y(n)|,$$

where $\tilde{\tau}_y(p^k) = 1$ if $|\tau(p)| \leq 1/(\log\log p)$ or $p > y$, and $\tilde{\tau}_y(p^k) = \tau(p^k)p^{-11k/2}$ otherwise.

Erdős-Kac type theorem for $(\log|\tau(n+a_1)|, \ldots, \log|\tau(n+a_k)|)$

Theorem (Klurman-M., '20+)

Let $k \geq 1$. If $\boldsymbol{a}$ is admissible then

$$\frac{1}{X}\left|\left\{n \leq X : n+a_j \in \mathcal{N}_\tau, \frac{\log|\tilde{\tau}_y(n+a_j)| + \frac{1}{2}\log\log X}{\sqrt{(1+\pi^2/6)\log\log X}} \in I_j \ \forall j\right\}\right|$$

$$= (2\pi)^{-k/2}\int_{I_1}\cdots\int_{I_k} e^{-\frac{1}{2}\|\boldsymbol{u}\|^2}\, d\boldsymbol{u} + o_{X\to\infty}(1),$$

where $\|\boldsymbol{u}\|^2 := \sum_j u_j^2$.

- proof uses the moment method
- case $k=1$ due to Luca, Radziwiłł and Shparlinski
- $\log|\tau(n+a_j)|$ (suitably normalized) are roughly independent Gaussians, and $1/k!$ is probability of independent Gaussians $X_1, \ldots X_k$ satisfying $X_1 < \cdots < X_k$