Averages of coefficients of $\mathrm{GL}_3 \times \mathrm{GL}_2$ *L*-functions

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Equidistribution of Hecke eigenvalues

Let $(\lambda_F(n))_{\geq 1}$ be a sequence of arithmetic function.
How uniform $\lambda_F(n)$'s are distributed in arithmetic progressions $n \equiv a \bmod q$? Do we have

$$\sum_{\substack{n \leq X \\ n \equiv a \bmod q}} \lambda_F(n) - \frac{1}{\phi(q)} \sum_{\substack{n \leq X \\ (n,q)=1}} \lambda_F(n) \ll_A \frac{X}{q} (\log X)^{-A} \qquad (1)$$

for $q \leq X^\vartheta$, with ϑ as large as possible? Here ϕ denotes Euler's phi function and the exponent ϑ is called the "*level of distribution*".

We are interested in the case where $\lambda_F(n)$'s are *Hecke eigenvalues* of an automorphic form F on $\mathrm{GL_d}$. By using Deligne's estimates for hyper-Kloosterman sum $Kl_d(an; q)$, one can take $\vartheta = \frac{2}{d+1} - \varepsilon$.
e.g. if $F \in \mathrm{GL}_3$

- for F the $1 \oplus 1 \oplus 1$ Eisenstein series, then $\lambda_F(n) = \tau_3(n)$, and $\vartheta = 1/2 + \eta$, results by Friedlander–Iwaniec, Heath-Brown, Fouvry–Kowalski–Michel, P. Xi, etc;
- for $F = 1 \oplus f$, $f \in \mathrm{GL}_2$, $\vartheta = 1/2 + \eta$, Kowalski–Michel–Sawin.

Sum of Hecke eigenvalues

In this talk, we study *analytic counterpart* of (1):
How small can one allow η to be:

$$\sum_{X \le n \le X + X^\eta} \lambda_F(n) = Res_{s=1} \frac{L(s, F)}{s} X^\eta + o(X^\eta);$$

equivalently, how small can the error term

$$A_d(X, F) - Res_{s=1} \frac{L(s, F)}{s} X = O(X^\eta)$$

be? Here

$$A_d(X, F) := \sum_{n \le X} \lambda_F(n).$$

What do we know about $A_d(X, F)$?
"Naive" error term:

$$A_d(X, F) - Res_{s=1} \frac{L(s, F)}{s} X = O(X^{1-\varepsilon}).$$

“trivial” bound for $A_d(X, F)$

A general result of Friedlander–Iwaniec (2005) states

$$A_d(X, F) - Res_{s=1} \frac{L(s, F)}{s} X = O(X^{\frac{d-1}{d+1}+\varepsilon}).$$

$$GRH \Rightarrow O(X^{1/2+\varepsilon}).$$

Question: How to surpass the exponent $\frac{d-1}{d+1}$?
We consider the case $F = \varphi \otimes f \in \mathrm{GL}_6$, where φ is a GL_3 cusp form and f is a GL_2 cusp form.

- Let $\{\lambda_\varphi(r, m)\}_{r,m\geq 1}$ and $\{\lambda_f(m)\}_{m\geq 1}$ be Fourier–Whittaker coefficients of φ and f. The Rankin–Selberg L-functions

$$L(s, \varphi \otimes f) := \sum_{r,m\geq 1} \frac{\lambda_\varphi(r, m)\lambda_f(m)}{(r^2 m)^s}.$$

Then $\lambda_F(n) := \sum_{r^2 m=n} \lambda_\varphi(r, m)\lambda_f(m)$ are coefficients of $L(s, \varphi \otimes f)$.

Main result: $F = \varphi \otimes f \in \mathrm{GL}_6$

Restrict F to $\varphi \otimes f \in \mathrm{GL}_3 \times \mathrm{GL}_2$, then Friedlander-Iwaniec implies

$$A_6(X, \varphi \otimes f) - Res_{s=1} \frac{L(s, \varphi \otimes f)}{s} X = O(X^{\frac{6-1}{6+1}+\varepsilon}).$$

Theorem (L.–Q. Sun, 2019)

For any $\delta < 1/364$, one has

$$A_6(X, \varphi \otimes f) = O(X^{\frac{5}{7}-\delta}),$$

Question: How should one proceed?

A *general* identity of Friedlander–Iwaniec

Recall

$$A_d(X,F) := \sum_{n\le X} \lambda_F(n).$$

Friedlander–Iwaniec observed that the study of $A_d(X,F)$ can be transformed to its "dual sum" involving $\overline{\lambda_F(n)}$:

$$\sum_{n\sim N} \overline{\lambda_F(n)}\, e(d\,(nX)^{1/d}) V\left(\frac{n}{N}\right), \quad e(x) := e^{2\pi i x}.$$

More precisely, by applying functional equation, they obtained

$$A_d(X,F) = \mathrm{Res}_{s=1} \frac{L(s,F)}{s} X + c_F\, X^{\frac{d-1}{2d}} B(X,N) + O\left(N^{-\frac{1}{d}} X^{\frac{d-1}{d}+\varepsilon}\right), \tag{2}$$

where

$$B(X,N) := \sum_{n\le N} \overline{\lambda_F(n)}\, n^{-\frac{d+1}{2d}} \cos(2\pi d\,(nX)^{1/d}).$$

Friedlander–Iwaniec's *general* error term

Estimating $B(X,N) \approx N^{-\frac{d+1}{2d}} \sum_{n\le N} \overline{\lambda_F(n)}\, cos(2\pi d\,(nX)^{1/d})$ trivially by $\ll N^{-\frac{d+1}{2d}} \cdot N$, and choosing N appropriately, F–I obtained

Proposition (Friedlander–Iwaniec, 2005)

Under Ramanujan–Selberg,

$$A_d(X,F) - \mathrm{Res}_{s=1} \frac{L(s,F)}{s} X = O(X^{\frac{d-1}{d+1}+\varepsilon}).$$

Remark

Landau's lemma also provides a similar result (provided the coefficients $\lambda_F(n)$'s are non-negative).

Conjectural *optimal* error term

Recall

$$B(X,N) = \sum_{n\le N} \overline{\lambda_F(n)}\, n^{-\frac{d+1}{2d}} \cos(2\pi d\,(nX)^{1/d}).$$

F–I conjectured $B(x,N) = O((xN)^{\varepsilon})$, correspondingly

Conjecture (Friedlander–Iwaniec, 2005)

One can take

$$A_d(X,F) - \mathrm{Res}_{s=1}\frac{L(s,F)}{s}X = O(X^{\frac{d-1}{2d}+\varepsilon}).$$

To improve over the error term $O(X^{\frac{d-1}{d+1}})$, it suffices to beat (for $N \approx X^{\frac{d-1}{d+1}}$)

$$\sum_{n\le N} \overline{\lambda_F(n)}\, e(d\,(nX)^{1/d}) = O(N).$$

Analytic twists of $\mathrm{GL}_3 \times \mathrm{GL}_2$

Now restrict to $d = 6$ and $\lambda_{\varphi \otimes f}(n) \in \mathrm{GL}_3 \times \mathrm{GL}_2$. Set

$$S_V(N, \varphi \otimes f) := \sum_{r,m \geq 1} \lambda_\varphi(r,m)\lambda_f(m) e(T\phi(\frac{r^2 m}{N})) V(\frac{r^2 m}{N}).$$

Theorem (L.–Q. Sun, 2019)

Let ϕ be either $\phi(x) = \log x$ or $\phi(x) = x^\beta$ (with $0 < \beta \leq 1/3$). One has

$$S_V(N, \varphi \otimes f) \ll T^{3/5+\varepsilon} N^{3/4}.$$

- Nontrivial (i.e., $= o(N)$) if $T^{3-3/5} \ll N$.
- **Rmk.** If we consider *algebraic twists*: $K =$ trace function modulo q, then [L.–Michel–Sawin, 2019]

$$\sum_{r,m \geq 1} \lambda_\varphi(r,m)\lambda_f(m) K(m) V(\frac{r^2 m}{N}) \ll N^{1-\delta}$$

provided $q^{3-1/4} \ll N$, under some assumptions on K.

Subconvexity of $\mathrm{GL}_3 \times \mathrm{GL}_2$ L-functions: t-aspect

Such a problem was first studied by R. Munshi for $\phi(x) = \log x$:

Theorem (R. Munshi, 2018)

$$\sum_{n=1}^{\infty} \lambda_\varphi(r,n)\lambda_f(n)n^{-iT} V\Big(\frac{r^2 n}{N}\Big) \ll r^{-1} T^{59/42+3\eta/2} N^{1/2+\varepsilon},$$

as long as $T^{3-\eta} \ll N$.

This implies (via *Approximate Functional Eq.*)

Corollary (R. Munshi, 2018)

$$L(1/2+iT, \varphi \otimes f) \ll T^{3/2-1/42+\varepsilon}.$$

Our refined estimate implies

Corollary (L.–Q. Sun, 2019)

$$L(1/2+iT, \varphi \otimes f) \ll T^{3/2-3/20+\varepsilon}.$$

Munshi's proof

Munshi uses

- Duke–Friedlander-Iwaniec δ-symbol:

  $$\delta(n-m,0)=\frac{1}{Q}\sum_{1\leq q\leq Q}\frac{1}{q}\sum_{a(q)}^{\star}e\Big(\frac{(n-m)a}{q}\Big)\int_{\mathbb{R}}g(q,x)e\Big(\frac{n-m}{qQ}x\Big)\mathrm{d}x;$$

  plus a "conductor-decreasing" trick:

  $$\frac{1}{K}\int_{\mathbb{R}}V\Big(\frac{v}{K}\Big)\Big(\frac{n}{m}\Big)^{iv}\mathrm{d}v$$

  that restricts $|n-m|<N/K$.
- GL_2 and GL_3-Voronoi summations;
- Cauchy–Schwarz;
- Poisson summation;
- Stationary phase + *third* derivative test of oscillatory integral.

Key steps of *our* proof

Our proof uses

- Write
$$\sum_{m\sim N}\lambda_\varphi(r,m)\lambda_f(m)e(T\phi(\frac{r^2m}{N}))=\sum_{m\sim N}\lambda_\varphi(r,m)\sum_{n\sim N}\lambda_f(n)e(T\phi(\frac{r^2n}{N}))\delta(n-m);$$
- DFI δ-symbol
$$\delta(n-m)=\frac{1}{Q}\sum_{1\leq q\leq Q}\frac{1}{q}\sum_{a(q)}^{\star}\underbrace{e\Big(\frac{(n-m)a}{q}\Big)}_{arithmetic}\int_{\mathbb{R}}g(q,x)\,\underbrace{e\Big(\frac{n-m}{qQ}x\Big)}_{archimedean}\,\mathrm{d}x;$$

  Rmk: The conductor decreasing effect has been built in the expression, an observation due to K. Aggarwal who removed the "conductor-decreasing" trick in Munshi's treatment in bounding $L(1/2+iT,\varphi)\ll T^{3/4-\delta}$, $\varphi\in\mathrm{GL}_3$.
- GL_2 and GL_3-Voronoi summations;
- Cauchy–Schwarz;
- Poisson summation;
- Stationary phase + *second* derivative test (two-dim'l version).

Two other examples

- $F = 1 \boxplus f \in \mathrm{GL}_3$, i.e., $\lambda_F(n) := \sum_{\ell m = n} \lambda_f(m)$:

Theorem (B. Huang–L.–Z. Wang, 2020)

Assuming $f \in \mathrm{GL}_2$ is holomorphic, one has

$$A_3(X, 1 \boxplus f) = L(1, f)X + O(X^{1/2-\delta}),$$

for any $\delta < 4/739$.

- $F = f \times f \in \mathrm{GL}_4$:

Theorem (B. Huang, 2020)

For any $\delta < 1/560$, one has

$$\sum_{n \leq X} \lambda_f(n)^2 = c_f X + O(X^{3/5-\delta}).$$

Common feature (in all 3 cases): the coefficients $\lambda_F(n)$ have **factorization**, making improvements possible!

Several questions

We recall

$$A_d(X,F) := \sum_{n\le X} \lambda_F(n) \underset{FI}{=} MT + O(X^{\frac{d-1}{d+1}}).$$

Questions:

- Let f be a GL_2 cusp form, how to improve the bound

$$A_2(X,f) = O(X^{1/3})?$$

  An upper bound $O(\frac{X^{1/3}}{\log^\gamma X})$ is known.
- For φ a GL_3 cusp form, how to improve

$$A_3(X,\varphi) = O(X^{1/2})?$$

  Rmk: We have a similar barrier for the "level of distribution" case: establish equidistribution of $\lambda_\varphi(1,n)$ in arithmetic progressions $n \equiv a \bmod q$ for $q \le X^\vartheta$ with $\vartheta = 1/2 + \eta$;
- How to get a "level of distribution" $\vartheta = 2/7 + \eta$ for $\mathrm{GL}_3 \times \mathrm{GL}_2$-coefficients?

Thank you !