Spectral Reciprocity via Integral representations

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What is (automorphic) spectral reciprocity?

Very roughly:

$$\sum_{\pi \in \mathcal{F}} \mathcal{L}(\pi)\mathcal{H}(\pi) = \sum_{\pi \in \widetilde{\mathcal{F}}} \widetilde{\mathcal{L}}(\pi)\widetilde{\mathcal{H}}(\pi),$$

where

- $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are families of automorphic representations,
- $\mathcal{L}(\pi)$ and $\widetilde{\mathcal{L}}(\pi)$ are some L-values associated to π,
- $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are some weight functions and
- this is not a direct consequence of the functional equation for the L-functions associated to $\mathcal{L}(\pi)$.

Maybe better explained with an example...

An example

Theorem (Blomer and Khan, 2017)

Let Π *be an unramified automorphic representation for* $\mathrm{GL}_3(\mathbb{Q})$*.*
Let q *and* ℓ *be coprime integers and let* $s, w \in \mathbb{C}$ *such that* $\frac{1}{2} \leq \mathrm{Re}(s) \leq \mathrm{Re}(w) < \frac{3}{4}$*. Then if*

$$\mathcal{M}(s, w, q, \ell, h) = \sum_{\mathrm{cond}(\pi)=q} \frac{L(s, \Pi \times \pi) L(w, \pi)}{L(1, Ad, \pi)} \frac{\lambda_\pi(\ell)}{\ell^w} h(\pi_\infty) + (\cdots),$$

we have

$$\mathcal{M}(s, w, q, \ell, h) = \mathcal{M}(s', w', \ell, q, \tilde{h}) + \textit{simple polar terms},$$

where $(s', w') = (\frac{1+w-s}{2}, \frac{3s+w-1}{2})$*,* $\tilde{h}$ *is an integral transform of* h*.*

What is it good for?

Application (Blomer and Khan, 2017)

Bounds for twisted moments

$$\sum_{\mathrm{cond}(\pi)=q} L(1/2,\pi)^4 \lambda_\pi(\ell)$$

and consequently, a subconvexity bound

$$L(1/2,\pi) \ll_\varepsilon \mathrm{cond}(\pi)^{\frac{1}{4}-\frac{1-2\vartheta}{24}+\varepsilon}$$

The method *etc.*

The method of Blomer and Khan is classical. It uses the Kuznetsov and Voronoi summation formulae and ultimately relies on the reciprocity identity

$$e\left(\frac{\overline{r}n}{c}\right)e\left(\frac{\overline{c}n}{r}\right)=e\left(\frac{n}{rc}\right).$$

Our Goal: Generalize Blomer-Khan's result to number fields.

It is often the case that an approach based on "classical" methods such as Kuznetsov and Voronoi stop working or at least become very cumbersome when generalized to number fields. Instead we use adelic methods and the theory of integral representations of L-functions.

Our main result

Theorem (N., 2020)

Let F be a number field and let $\mathfrak{q}$ and $\mathfrak{l}$ be coprime ideals. Let Π be an unramified cuspidal representation for $\mathrm{GL}_3(F)$ and let $s, w, s', w' \in \mathbb{C}$ be as in Blomer-Khan's theorem. Then if

$$\mathcal{M}_0(s, w, \mathfrak{q}, \mathfrak{l}) = \frac{1}{N\mathfrak{q}} \sum_{\substack{\mathrm{cond}(\pi)=q \\ \pi_\infty \text{ spherical}}} \frac{\Lambda(s, \Pi \times \pi)\Lambda(w, \pi)}{\Lambda(1, Ad, \pi)} \frac{\lambda_\pi(\mathfrak{l})}{(N\mathfrak{l})^w} + (\cdots),$$

we have

$$\mathcal{M}_0(s, w, \mathfrak{q}, \mathfrak{l}) = \mathcal{M}_0(s', w', \mathfrak{l}, \mathfrak{q}) + \text{simple polar terms.}$$

Our result *vs.* Blomer-Khan's

Our result

1. works for general number fields,
2. does not allow for general weight functions. In particular, our sums is supported on representations that are spherical at the archimedean places,
3. requires the $GL(3)$ form to be *cuspidal*. In particular we cannot deduce a generalization to BK's subconvexity estimate.

A non-vanishing result

Corollary (N., 2020)

Let F be number field and let Π be an unramified cuspidal automorphic representation for $\mathrm{GL}_3(F)$. Then for prime ideals $\mathfrak{p}$ of F with sufficiently large norm, there is at least one automorphic cuspidal automorphic representation π, such that π_∞ is spherical and $\mathrm{cond}(\pi) = \mathfrak{p}$ and

$$L(1/2, \Pi \times \pi) L(1/2, \pi) \neq 0.$$

This generalizes[1] a non-vanishing result of R. Khan over $\mathbb{Q}$.
As a matter of fact, the argument in Khan's paper is substantially similar to that of Blomer-Khan.

[1] up to the restriction on sphericity

Proof Sketch

The theory of integral representations as developed by Jacquet–Piatestski-Shapiro–Shalika states that for unramified automorphic representations Π and π of $\mathrm{GL}(3)$ and $\mathrm{GL}(2)$ respectively, there exist vectors $\Phi \in \Pi$ and $\phi \in \pi$ such that

$$\Lambda(s, \Pi \times \pi) = \int_{\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} \Phi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \phi(h) \, |\mathrm{det} h|^{s-1/2} \, \mathrm{d}h$$

and

$$\Lambda(w, \pi) = \int_{F^\times \backslash \mathbb{A}^\times} \phi \begin{pmatrix} y & \\ & 1 \end{pmatrix} |y|^{s-1/2} \mathrm{d}^\times y,$$

for $\mathrm{Re}(s)$ and $\mathrm{Re}(w)$ sufficiently large. The so-called spherical vectors.

First case: $\mathfrak{q} = \mathfrak{l} = 1$

We let

$$\mathcal{A}_s\Phi(h) := |\det h|^{s-1/2} \int_{F^\times\backslash\mathbb{A}^\times} \begin{pmatrix} z(u)h & \\ & 1 \end{pmatrix} |u|^{2s-1} \mathrm{d}^\times u,$$

where $z(u) = \begin{pmatrix} u & \\ & u \end{pmatrix}$ and spectrally decomposing the period

$$\mathcal{P}(s, w, \Phi) = \int_{F^\times\backslash\mathbb{A}^\times} \mathcal{A}_s\Phi \begin{pmatrix} y & \\ & 1 \end{pmatrix} |y|^{w-1/2} \mathrm{d}^\times y$$

we arrive at

$$\sum_\pi \langle \mathcal{A}_s\Phi, \phi_\pi \rangle \int \phi_\pi \begin{pmatrix} y & \\ & 1 \end{pmatrix} |y|^{w-1/2} \mathrm{d}^\times y =$$

$$= \sum_{\pi \text{ spherical}} L(s, \Pi \times \pi) L(w, \pi) + (\cdots) \quad (1)$$

Period reciprocity

We unfold $\mathcal{P}(s, w, \Phi)$ as

$$\int_{F^\times\backslash\mathbb{A}^\times}\int_{F^\times\backslash\mathbb{A}^\times}\Phi\begin{pmatrix} uy & & \\ & u & \\ & & 1\end{pmatrix}|u|^{2s-1}|y|^{s+w-1}\mathrm{d}^\times u\mathrm{d}^\times y.$$

Now we look at the identity

$$\begin{pmatrix} uy & & \\ & u & \\ & & 1\end{pmatrix} = \begin{pmatrix} u & & \\ & u & \\ & & u\end{pmatrix} w_{23}\begin{pmatrix} y & & \\ & u^{-1} & \\ & & 1\end{pmatrix} w_{23},$$

where w_{23} is the Weyl element corresponding to the permutation $(2,3)$.

One last step

Since Φ is left invariant by $Z_3(\mathbb{A})\mathrm{GL}_3(F)$ and right invariant by $\mathrm{GL}_3(\mathfrak{o}_F)$, we have

$$\Phi\begin{pmatrix} uy & & \\ & u & \\ & & 1 \end{pmatrix} = \Phi\begin{pmatrix} y & & \\ & u^{-1} & \\ & & 1 \end{pmatrix}.$$

Replacing this in the definition of $\mathcal{P}(s,w,\Phi)$, we get

$$\mathcal{P}(s,w,\Phi) = \int_{F^\times\backslash\mathbb{A}^\times}\int_{F^\times\backslash\mathbb{A}^\times} \Phi\begin{pmatrix} y & & \\ & u^{-1} & \\ & & 1 \end{pmatrix} |u|^{2s-1}|y|^{s+w-1}\mathrm{d}^\times u\mathrm{d}^\times y.$$

Now, changing variables we see that $\mathcal{P}(s,w,\Phi) = \mathcal{P}(s',w',\Phi)$ and the reformula holds.

What about higher level?

Finally, for larger conductor, we introduce matrices of the shape

$$u_\beta = \begin{pmatrix} 1 & & \beta q^{-1} \\ & 1 & \\ & & 1 \end{pmatrix} \text{ and } v_\beta = \begin{pmatrix} 1 & \beta q^{-1} & \\ & 1 & \\ & & 1 \end{pmatrix}$$

The first one acts as a level-raising operator and the second is related to the Hecke operators.
Finally, the same change of variable as before intechanges the action of these matrices since

$$u_\beta = w_{23} v_\beta w_{23}.$$

This obsevation along with local computations by Kim (2010) and Booker-Krishnamurthy-Lee (2019) lead us to the main formula.

Thank You!