Nonvanishing for cubic L-functions

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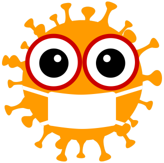

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Moments of Dirichlet L functions

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}=\prod_{p \text{ prime}}\left(1-\frac{\chi(p)}{p^s}\right)^{-1} \qquad \mathrm{Re}(s)>1,$$

$$\sum_{\substack{\chi \bmod d\\ d\leq X}}^{*} L(\tfrac{1}{2},\chi)^k,$$

where $*$ indicates "primitive characters" associated to nonvanishing results.

Chowla's conjecture: $L(\frac{1}{2},\chi)\neq 0$ for Dirichlet L-functions associated to primitive characters χ.

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Nonvanishing results from moments

By Cauchy–Schwarz,

$$\sideset{}{^*}\sum_{\substack{\chi \bmod d\\ d\le X}} L(\tfrac{1}{2},\chi) \le \left(\sideset{}{^*}\sum_{\substack{\chi \bmod d\\ d\le X}} L\left(\tfrac{1}{2},\chi\right)^2\right)^{1/2} \left(\sideset{}{^*}\sum_{\substack{\chi \bmod d\\ d\le X\\ L\left(\frac{1}{2},\chi\right)^2\neq 1}} 1\right)^{1/2}$$

$$\#\left\{\chi \bmod d, d\le X, L\left(\frac{1}{2},\chi\right)\neq 0\right\}^* \ge \frac{\left(\sideset{}{^*}\sum_{\substack{\chi \bmod d\\ d\le X}} L\left(\frac{1}{2},\chi\right)\right)^2}{\sideset{}{^*}\sum_{\substack{\chi \bmod d\\ d\le X}} L\left(\frac{1}{2},\chi\right)^2}.$$

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Moments of primitive quadratic Dirichlet L-functions

Using Random Matrix Theory, Keating and Snaith (2000) conjectured that

$$\sum_{\substack{\chi \bmod d \\ \text{quadratic}, d \leq X}}^{*} L(\tfrac{1}{2}, \chi)^k \sim c_k X (\log X)^{\frac{k(k+1)}{2}}.$$

- $k = 1$ Jutila (1981)
- $k = 2$ Jutila (1981), Soundararajan (secondary main term, 2000)
- $k = 3$ Soundararajan (2000), Diaconu, Goldfeld, Hoffstein (2003)
- $k = 4$ Shen (2019+ under GRH) following Soundararajan and Young (2010).

$$\text{nonvanishing} \gg \frac{X}{\log X} \sim X^{1-\varepsilon}.$$

- Soundararajan (2000) with mollified moments: nonvanishing for at least $7/8$.

Moments for cubic characters

- Conjecture (Conrey-Farmer-Keating-Rubinstein-Snaith (2005))

$$\sum^{*}_{\substack{\chi \bmod d \\ \text{cubic}, d \leq X}} L(\tfrac{1}{2}, \chi)^k \sim c_k X.$$

- $\mathbb{Q}$ (no Kummer case), $k = 1$ Baier & Young (2010)
- $\mathbb{Q}(\xi_3)$ (Kummer case, Hecke L functions), Luo (2004)

first moment $\sim cX$ for a thin family ($\square$-free conductor).

$$\#\{\chi \pmod{d} : \chi^3 = \chi_0, N(d) \leq X\}^* \sim CX \log X$$

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Function fields

Let q power of a prime, $\mathbb{F}_q$ finite field with q elements.

Number Fields		Function Fields
$\mathbb{Q}$	$\leftrightarrow$	$\mathbb{F}_q(T)$
$\mathbb{Z}$	$\leftrightarrow$	$\mathbb{F}_q[T]$
p positive prime	$\leftrightarrow$	$P(T)$ monic irreducible polynomial
$\|n\| = \|\mathbb{Z}/n\mathbb{Z}\| = n \in \mathbb{N}$	$\leftrightarrow$	$\|F(T)\| = \|\mathbb{F}_q[T]/(F(T))\| = q^{\deg F}$
$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$	$\leftrightarrow$	$\zeta_q(s) = \sum_{\substack{F \in \mathbb{F}_q[T] \\ F \text{ monic}}} \frac{1}{\|F\|^s}$
Riemann Hypothesis ☹	$\leftrightarrow$	Riemann Hypothesis ☺

Moments of quadratic L-functions over function fields

Andrade and Keating (2014) conjectured

$$\sum_{\substack{\chi \text{ quadratic} \\ \text{genus}(\chi)=g}}^{*} L(\tfrac{1}{2},\chi)^k \sim c_k q^{2g+1}(2g+1)^{\frac{k(k+1)}{2}}$$

and proved this for $k = 1$.

- Florea (2017, several papers) second order term for $k = 1$ and cases $k = 2, 3, 4$.
- Bui and Florea (2016) nonvanishing for $\geq 94\%$ using one-level density, studying zeros near $1/2$.
- Li (2018) vanishing $\gg (q^{2g+1})^{\frac{1}{3}-\varepsilon}$.

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First moments of cubic L-functions over function fields

Theorem (David, Florea, L. (2019+))

Let q be an odd prime power such that $q \equiv 2 \pmod 3$. Then

$$\sideset{}{^*}\sum_{\substack{\chi \text{ cubic} \\ \text{genus}(\chi)=g}} L(\tfrac{1}{2}, \chi) = A q^{g+2} + O\left(q^{\frac{7g}{8}+\varepsilon g}\right).$$

Theorem (David, Florea, L. (2019+))

Let q be an odd prime power such that $q \equiv 1 \pmod 3$. Let χ_3 be a fixed cubic character on $\mathbb{F}_q^*$

$$\sideset{}{^*}\sum_{\substack{\chi \text{ cubic} \\ \text{genus}(\chi)=g \\ \chi|_{\mathbb{F}_q^*}=\chi_3}} L(\tfrac{1}{2}, \chi) = C_1 g q^{g+1} + C_2 q^{g+1} + O\left(q^{g\frac{1+\sqrt{7}}{4}+\varepsilon g}\right).$$

Nonvanishing for cubic L-functions (Non Kummer)

- nonvanishing for $\gg q^g/g^{1+\varepsilon}$.
- Ellenberg-Li-Shusterman (2020) nonvanishing for $\gg q^g/g^{1/2}$.
- One-level density does not give a positive proportion for nonvanishing.

Theorem (David, Florea, L. (2020+))

Let q be an odd prime power such that $q \equiv 2 \pmod 3$.

$$\# \left\{ \chi \text{ cubic}, \text{genus}(\chi) = g, L\left(\frac{1}{2}, \chi\right) \neq 0 \right\}^* \geq cq^g,$$

where $c > 0$ is an explicit constant.

General Strategy - Mollified moments

Idea goes back to Selberg (1946) and Soundararajan (2000)

$$M(\chi) \approx \sum_{\substack{P|f \Rightarrow \deg P \leq N \\ \Omega(f) \leq \ell}} \frac{\lambda(f)\chi(f)}{|f|^{\frac{1}{2}}}, \text{ behaves "like } L(\tfrac{1}{2},\chi)^{-1}\text{"}.$$

$$\sum_{\substack{\chi \text{ cubic} \\ \text{genus}(\chi)=g}}^{*} L(\tfrac{1}{2},\chi)M(\chi) \sim B_1 q^g,$$

$$\sum_{\substack{\chi \text{ cubic} \\ \text{genus}(\chi)=g}}^{*} |L(\tfrac{1}{2},\chi)M(\chi)|^2 \leq B_2 q^g.$$

By Cauchy–Schwarz

$$\#\left\{\chi \text{ cubic}, \text{genus}(\chi) = g, L\left(\frac{1}{2},\chi\right) \neq 0\right\}^* \geq \frac{B_1^2}{B_2} q^g.$$

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The first mollified moment

Evaluate

$$\sum_{\substack{\chi \text{ cubic} \\ \text{genus}(\chi)=g}}^{*} L(\tfrac{1}{2},\chi)\chi(h),$$

where h is a polynomial (of low degree) coming from the mollifier.

- Approximate functional equation gives principal and dual terms.
- Main term coming from cubes in principal term, Perron formula.
- Non-cubes from principal term bounded by Lindelöf.
- Dual term coming from generating series of cubic Gauss sums.

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Bounding the mollified second moment

Our goal:

$$\sum_{\substack{\chi \text{ cubic} \\ \text{genus}(\chi)=g}}^{*} |L(\tfrac{1}{2},\chi)M(\chi)|^2 \leq B_2 q^g.$$

The method for finding this bound depends on works of

- Soundararajan (2009): almost sharp bounds for moments of $\zeta(s)$ (under RH).
- Harper (2013+): sharp bounds for moments of $\zeta(s)$ (under RH).
- Lester & Radziwiłł (2019+): sharp bounds for mollified moments in the family of quadratic twists of modular forms (under GRH).

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Bounding L

Soundararajan (2009), adapted by Bui, Florea, Keating, Roditty-Gershon (2019) to function fields:
Bounding $\log|L(\frac{1}{2},\chi)|$ by a short Dirichlet polynomial.

$$\log|L(\tfrac{1}{2},\chi)| \leq \sum_{\deg(P)\leq N} \frac{\mathrm{Re}(\chi(P))a(P)}{|P|^{\frac{1}{2}}} + \frac{g}{N} + O(1).$$

Say $N \approx \frac{g}{e^{90}}$,

$$|L(\tfrac{1}{2},\chi)|^k \leq \exp\left(k \sum_{\deg P\leq N} \frac{\mathrm{Re}(\chi(P))a(P,N)}{|P|^{\frac{1}{2}}} + \text{small things}\right).$$

Bounding the exponential

$$|L(\tfrac{1}{2},\chi)|^k \le \exp\left(k \sum_{\deg P \le N} \frac{\mathrm{Re}(\chi(P))a(P,N)}{|P|^{\frac{1}{2}}} + \text{small things}\right).$$

Lester & Radziwiłł (2019+): Use the bound

$$e^t \le (1+e^{-\ell/2}) \sum_{s \le \ell} \frac{t^s}{s!}$$

for $t \le \ell/e^2$ and ℓ even.

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How this leads to the mollifier

Sums of

$$\frac{(\mathrm{Re}\, P_I(\chi))^s}{s!} = \frac{1}{2^s} \sum_{\substack{P|fh \Rightarrow P \in I \\ \Omega(fh)=s}} \frac{\chi(fh^2)a(fh)\nu(f)\nu(h)}{|fh|^{\frac{1}{2}}},$$

where $\nu(P^a) = \frac{1}{a!}$ multiplicative, Ω number of prime factors counted with multiplicity.
This leads to

$$M(\chi) = \sum_{\substack{P|f \Rightarrow P \in I \\ \Omega(f) \leq \ell}} \frac{\lambda(f)\chi(f)\nu(f)}{|f|^{\frac{1}{2}}},$$

where $\lambda(n) = (-1)^{\Omega(n)}$ is the Liouville function.

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Bounding the mollified moment with the exponential

Say $I = (0, Ag]$.

$$\sum_{\substack{\chi \text{ cubic} \\ \text{genus}(\chi)=g}}^{*} (\mathrm{Re}\, P_I(\chi))^s |M(\chi)|^k \ll q^g g^{O(1)} (C(\log g)^{1/2} A)^{1/A}$$

It does not work if A is a constant, take $A \sim \frac{1}{\log g}$.

$$\ll \frac{q^g}{(\log g)^{(\log g)/2}} \ll \frac{q^g}{g^{100000000}}.$$

Following Lester & Radziwiłł (2019+),

$$I_0 = (0, g\theta_0], \quad I_1 = (g\theta_0, g\theta_1], \ldots, I_J = (g\theta_{J-1}, g\theta_J],$$

$$\theta_j = \frac{e^j}{(\log g)^{1000}}.$$

Other intervals

$$\leq e^{e^{O(k)}} q^g.$$

Bounding the mollified moment in the other cases

$$\sum^*_{\substack{\chi \text{ cubic} \\ \text{genus}(\chi)=g \\ \mathrm{Re}\, P_I(\chi) > \frac{\ell}{ke^2}}} |L(\tfrac{1}{2},\chi)|^k |M(\chi)|^k$$

$$\leq \sum^*_{\substack{\chi \text{ cubic} \\ \text{genus}(\chi)=g}} |L(\tfrac{1}{2},\chi)|^k |M(\chi)|^k \underbrace{\left(\frac{ke^2\, \mathrm{Re}\, P_I(\chi)}{\ell}\right)^s}_{\geq 1}$$

$$\leq \underbrace{\Bigg(\sum^*_{\substack{\chi \text{ cubic} \\ \text{genus}(\chi)=g}} |L(\tfrac{1}{2},\chi)|^{2k}\Bigg)^{1/2}}_{(q^g g^{k^2})^{1/2}} \underbrace{\Bigg(\sum^*_{\substack{\chi \text{ cubic} \\ \text{genus}(\chi)=g}} \left(\frac{ke^2}{\ell}\right)^{2s} |M(\chi)|^{2k} (\mathrm{Re}\, P_I(\chi))^{2s}\Bigg)^{1/2}}_{\left(\frac{q^g}{(\log g)^{(\log g)/2}}\right)^{1/2}}$$

$$= o(q^g)$$

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An explicit constant (somehow small)

Theorem (David, Florea, L. (2020+))

Let q be an odd prime power sucht that $q \equiv 2 \pmod 3$. Then

$$\frac{\#\left\{\chi,\ \textit{cubic, genus}(\chi) = g,\ L(\tfrac{1}{2}, \chi) \neq 0\right\}^*}{\#\left\{\chi,\ \textit{cubic, genus}(\chi) = g\right\}^*} \geq 0.47 e^{-e^{182}}$$

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Merci de votre attention!!!

Thanks for your attention!!!