Gelfand's trick for the spherical derived Hecke algebra

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Motivation

Γ "nice" arithmetic group, e.g.:
- $\Gamma = \mathrm{SL}_2(\mathbb{Z}), \mathrm{SL}_3(\mathbb{Z}), \dots$
- $\Gamma = \mathrm{SL}_2(\mathcal{O}_K)$, K imaginary quadratic
- congruence subgroups, e.g., $\Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$

Γ acts on a symmetric space, e.g.:
- $\mathrm{SL}_2(\mathbb{Z})$ acts on the complex upper half plane
- $\Gamma = \mathrm{SL}_2(\mathcal{O}_K)$ acts on hyperbolic 3-space

Hecke algebra acts on $\mathrm{H}^i(\Gamma\backslash X, \mathbb{Q})$
Phenomenon: same Hecke eigensystem can occur in several degrees

Venkatesh $\rightsquigarrow$ global derived Hecke algebra $\mathcal{H}^*_{\mathbb{Q}_\ell}$:
- $\mathcal{H}^*_{\mathbb{Q}_\ell}$ is a graded $\mathbb{Q}_\ell$-algebra
- $\mathrm{H}^*(\Gamma\backslash X, \mathbb{Q}_\ell)$ is a graded $\mathcal{H}^*_{\mathbb{Q}_\ell}$-module

Conjecture: derived Hecke algebra "causes" the phenomenon

Setup

$F/\mathbb{Q}_p$ finite extension, $\mathcal{O}$ ring of integers, $\mathcal{O}/(\varpi) = \mathbb{F}_q$

$\mathbb{G}/\mathcal{O}$ split reductive group
$\rightsquigarrow G = \mathbb{G}(F)$, $K = \mathbb{G}(\mathcal{O})$, $T \subseteq G$ maximal split torus

Example: $\mathbb{G} = \mathrm{GL}_n$
$\rightsquigarrow G = \mathrm{GL}_n(F)$, $K = \mathrm{GL}_n(\mathcal{O})$, T= invertible diagonal matrices

Cartan decomposition (Bruhat-Tits): $G = KTK$
If $G = \mathrm{GL}_n(F)$, this is just the elementary divisor theorem

Classical spherical Hecke algebra

Spherical Hecke algebra of G:

$$\mathcal{H}(G)_{\mathbb{C}} = \mathbb{C}[K\backslash G/K] = C_c(K\backslash G/K, \mathbb{C})$$

Product given by convolution:

$$(h_1 * h_2)(g) = \int_G h_1(gx^{-1}) \cdot h_2(x)dx$$

$G = KTK \Rightarrow \{\mathbb{1}_{KtK} \mid t \in T\}$ generates $\mathcal{H}(G)_{\mathbb{C}}$

Gelfand's trick $\Rightarrow \mathcal{H}(G)_{\mathbb{C}}$ is commutative

Gelfand's trick for the spherical Hecke algebra

$\exists\ \sigma \colon G \to G$ involution such that

- $\sigma(K) = K$
- $\sigma(t) = t^{-1} \quad \forall t \in T$

If $G = \mathrm{GL}_n(F)$, take $\sigma(g) = {}^t g^{-1}$

Define $\sigma \colon \mathcal{H}(G)_{\mathbb{C}} \to \mathcal{H}(G)_{\mathbb{C}}$ by

$$\sigma(f)(g) = f(\sigma(g)^{-1})$$

Easy calculation $\rightsquigarrow \sigma(h_1 * h_2) = \sigma(h_2) * \sigma(h_1)$
Cartan decomposition $\Rightarrow \sigma(h) = h$

Derived Hecke algebra

$$\begin{aligned}\mathcal{H}(G)_{\mathbb{C}} &= \mathbb{C}\,[K\backslash G/K] \\ &= \mathrm{Hom}_K(\mathbb{C}, \mathbb{C}[G/K]) \\ &= \mathrm{Hom}_G(\mathbb{C}[G/K], \mathbb{C}[G/K])\end{aligned}$$

Idea:

$$\mathcal{H}^*(G)_{\mathbb{C}} := \bigoplus_i \mathrm{Ext}^i_G(\mathbb{C}[G/K], \mathbb{C}[G/K])$$

Problem:

$$\mathrm{Ext}^i_G(\mathbb{C}[G/K], \mathbb{C}[G/K]) = 0 \text{ for } i \geq 1$$

Solution: choose $\ell \neq p$ and consider

$$\mathcal{H}^*(G)_{\mathbb{Z}/\ell^r\mathbb{Z}} := \bigoplus_i \mathrm{Ext}^i_G(\mathbb{Z}/\ell^r\mathbb{Z}[G/K], \mathbb{Z}/\ell^r\mathbb{Z}[G/K])$$

The main result

Theorem

$\ell \neq p$ prime such that

- $q \equiv 1 \mod \ell$ *and*
- *ℓ does not divide the order of the Weyl group of G*

$$\Rightarrow \mathcal{H}^*(G)_{\mathbb{Z}/\ell^r\mathbb{Z}} \text{ is graded-commutative for all } r$$

Weyl group of GL_n isomorphic to S_n
$\rightsquigarrow$ second condition equivalent to $\ell > n$ if $G = \mathrm{GL}_n(F)$

Result previously known under the condition $q \equiv 1 \mod \ell^r$ (Venkatesh)

An explicit model for the derived Hecke algebra

$(x,y) \in G/K \times G/K \rightsquigarrow G_{xy}$ its stabilizer in G

An element $h \in \mathcal{H}(G)^*_R$ is a collection of elements

$$h(x,y) \in \mathrm{H}^*(G_{xy}, R), \quad (x,y) \in G/K \times G/K$$

such that

- h is G-invariant, i.e. $\mathrm{Ad}(g)^* h(gx, gy) = h(x,y)$
- h has finite support modulo G

Multiplication given by convolution:

$$(h_1 * h_2)(x,z) = \sum_{y \in G_{xz} \backslash G/K} \mathrm{Cores}^{G_{xz}}_{G_{xyz}} (\mathrm{Res}^{G_{xy}}_{G_{xyz}} h_1(x,y) \cup \mathrm{Res}^{G_{yz}}_{G_{xyz}} h_2(y,z))$$

The involution

G=KTK $\Rightarrow$ (x, y) and $(\sigma(y), \sigma(x))$ lie in same G-orbit, i.e.:
$\exists\ g_{xy} \in G$ such that $g_{xy}(x, y) = (\sigma(y), \sigma(x))$

$$g_{xy} G_{xy} g_{xy}^{-1} = \sigma(G_{xy})$$
$$\rightsquigarrow \mathrm{Ad}(g_{xy})^* \sigma^* \colon\ \mathrm{H}^*(G_{xy}, R) \to \mathrm{H}^*(G_{xy}, R)$$

Define $h^\sigma \in \mathcal{H}(G)_R^*$ via $h^\sigma(x, y) = \mathrm{Ad}(g_{xy})^* \sigma^*(h(x, y))$

Easy calculation
$\rightsquigarrow \sigma(h_1 * h_2) = (-1)^{ij} \cdot \sigma(h_2) * \sigma(h_1)$, $\deg(h_1) = i$, $\deg(h_2) = j$

Under assumptions of theorem: $\sigma(h_1 * h_2) = \sigma(h_1) * \sigma(h_2)$

How to prove it?

Would you like to know more?
Check out my preprint!