Multiplicative functions in short intervals revisited

Kaisa Matomäki
(Joint work with M. Radziwiłł)

University of Turku, Finland

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What are multiplicative functions?

We say that $f\colon \mathbb{N} \to \mathbb{C}$ is multiplicative if

$$f(mn) = f(m)f(n) \quad \text{whenever } \gcd(m, n) = 1.$$

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- The Möbius function

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 \cdots p_k \text{ with } p_j \text{ distinct;} \\ 0 & \text{otherwise.} \end{cases}$$

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- The indicator function for the set $\mathcal{N}$ of numbers that can be written as a sum of two squares;

$$\mathbf{1}_{\mathcal{N}}(p^k) = \begin{cases} 0 & \text{if } p \equiv 3 \pmod 4 \text{ and } k \text{ is odd;} \\ 1 & \text{otherwise.} \end{cases}$$

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- The indicator function of the set of y-smooth numbers (n is y-smooth if $p \mid n \implies p \leq y$).

Averages over $n \leq x$

- Averages of multiplicative functions $f \colon \mathbb{N} \to [-1, 1]$ over $n \leq x$ are well understood (at least qualitatively):
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- Wirsing: If (1) does not hold, then
$$\frac{1}{x} \sum_{n \le x} f(n) = o(1); \qquad \text{e.g. } \frac{1}{x} \sum_{n \le x} \mu(n) = o(1).$$
- Halász's theorem gives quantitative results.

Short averages

- Radziwiłł and I have shown that the same story holds in almost all short intervals: For $f : \mathbb{N} \to [-1, 1]$, one has

$$\left| \frac{1}{h} \sum_{x < n \leq x+h} f(n) - \frac{1}{X} \sum_{X < n \leq 2X} f(n) \right| = O\left((\log h)^{-1/200} \right)$$

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- The theorem is trivial e.g. for $1_{n\in\mathcal{N}}$ (the indicator function of sums of two squares) since the long average is $C(\log X)^{-1/2}$.
- For many applications, one needs a result for complex f.

Sums of two squares in short intervals

- Recall $\mathcal{N}$ is the set of numbers that can be written as a sum of two squares. Then

$$\frac{1}{x}\sum_{n\leq x} 1_{\mathcal{N}}(n) = (C + o(1))\frac{1}{(\log x)^{1/2}}.$$

- This means that the average gap between consecutive $m, n \in \mathcal{N} \cap [X, 2X]$ is $\asymp (\log X)^{1/2} =: h_1$.

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- In particular in intervals of length $y = o(h_1)$, typically $\sum_{x<n\le x+y} 1_{\mathcal{N}}(n) = 0$.
- But for longer intervals one would expect regular behaviour, i.e

$$\left|\frac{1}{h_0 h_1}\sum_{x<n\le x+h_0 h_1} 1_{\mathcal{N}}(n) - \frac{C}{(\log X)^{1/2}}\right| = o\left(\frac{1}{(\log X)^{1/2}}\right)$$

for almost all $x \in (X, 2X]$ as soon as $h_0 \to \infty$ with $x \to \infty$

Sums of two squares in short intervals

Theorem (M-Radziwiłł (202?))

For any $\delta > 0$, and any $h_0 \geq 1$,

$$\left|\frac{1}{h_0(\log X)^{1/2}} \sum_{x<n\leq x+h_0(\log X)^{1/2}} 1_{\mathcal{N}}(n) - \frac{C}{(\log X)^{1/2}}\right| \leq \frac{\delta}{(\log X)^{1/2}}.$$

for all but at most

$$O_\delta(Xh_0^{-c\delta^{12}})$$

integers $x \in (X, 2X]$, for some $c > 0$.

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- Note that the exceptional set bound saves polynomially in h_0.
- Previously Hooley (1994) and Plaksin (1987, 1992) showed that, for almost all $x \in (X, 2X]$ one has

$$\frac{1}{h_0(\log X)^{1/2}} \sum_{x<n\leq x+h_0(\log X)^{1/2}} 1_{\mathcal{N}}(n) \asymp \frac{1}{(\log X)^{1/2}}.$$

General vanishing case

- When $|f(p)|$ has average value $\alpha \in (0,1)$, it is known that
$$\frac{1}{x}\sum_{n\leq x}|f(n)| \asymp \prod_{p\leq x}\left(1+\frac{|f(p)|-1}{p}\right) \asymp (\log x)^{\alpha-1}.$$
- Write $h_1 := \prod_{p\leq x}\left(1+\frac{|f(p)|-1}{p}\right)^{-1} \asymp (\log x)^{1-\alpha}$.

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$$\left|\frac{1}{h_0 h_1}\sum_{x<n\leq x+h_0 h_1} f(n) - \frac{1}{X}\sum_{X<n\leq 2X} f(n)\right|$$

$$= o\left(\prod_{p\leq X}\left(1+\frac{|f(p)|-1}{p}\right)\right)$$

for almost all x as soon as $h_0 \to \infty$ with $x \to \infty$

Short intervals, vanishing case

Theorem (M-Radziwiłł (202?))

Let $\varepsilon > 0$. *Let* $f : \mathbb{N} \to [-1, 1]$ *be a multiplicative function s.t.*

$$\sum_{w<p\leq z} \frac{|f(p)|}{p} \geq \varepsilon \sum_{w<p\leq z} \frac{1}{p} + O\left(\frac{1}{\log w}\right)$$

for all $2 \leq w < z < x^{\varepsilon}$. *Set* $h_1 := \prod_{p\leq X} \left(1 + \frac{1-|f(p)|}{p}\right)$.

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for all but at most $O(X h_0^{-c\delta^{\kappa}})$ *integers* $x \in (X, 2X]$, *for some* $c = c(\varepsilon)$ *and* $\kappa = \kappa(\varepsilon) > 0$.

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If f is complex-valued, a twist in main term.

Limitations of Hooley's and Plaksin's methods

- Recall Hooley's and Plaksin's works giving that for almost all x one has

$$\frac{1}{h_0(\log X)^{1/2}} \sum_{x<n\leq x+h_0(\log X)^{1/2}} 1_{\mathcal{N}}(n) \asymp \frac{1}{(\log X)^{1/2}}.$$

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- The main arithmetic information they used was the solution to the shifted convolution problem

$$\sum_{n\leq x} r_K(n) r_K(n+h) \tag{2}$$

with $r_K(n)$ the coefficients of the Dedekind zeta function of $K = \mathbb{Q}(i)$.

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- (2) is completely open when degree of K exceeds two. Hence the previous approaches completely fail for generalisations.
- We only use multiplicativity, so we have chances to generalise!

Norm forms

- We say an integer n is norm-form of a number field K if n equals a norm of an algebraic integer in K. Write $g_K(n)$ for the indicator function. In particular $g_{\mathbb{Q}(i)}(n) = 1_{n \in \mathcal{N}}(n)$

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- By Odoni's work the density in $[1, x]$ of norm forms of K is

$$\delta_K(x) := \prod_{\substack{p \leq x,\, p \neq N\mathfrak{a} \\ \mathfrak{a} \text{ integral ideal}}} \left(1 - \frac{1}{p}\right)$$

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- However, following work of Odoni, we show that $g_K(n)$ is a linear combination of (complex-valued) multiplicative functions.
- Applying our results to each function in the linear combination, we get a theorem in arbitrary number fields.

Norm forms in short intervals

Theorem (M-Radziwiłł (202?))

Let K be a number field over $\mathbb{Q}$, and let

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Then, as $X \to \infty$, uniformly in $2 \leq h \leq X$ one has

$$\left| \frac{1}{h\delta_K(X)^{-1}} \sum_{x \leq n \leq x + h\delta_K(X)^{-1}} g_K(n) - C_K \delta_K(X) \right| \leq \varepsilon \delta_K(X)$$

for all $x \in (X, 2X]$ with at most $O_\varepsilon(Xh^{-c\varepsilon^\kappa})$ exceptions where $c, \kappa, C_K > 0$ depend solely on K.

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This is a vast extension of Hooley's and Plaksin's works for $\mathbb{Q}(i)$, with an asymptotic formula.

Gaps between sums of two squares

- Hooley (1971) and Plaksin (1987, 1992) have also studied gaps between sums of two squares.
- Writing $1 = s_1 < s_2 < \dots$ for the sequence of integers in $\mathcal{N}$. Plaksin showed that, for any $\gamma \in [1, 2)$ (Hooley: $\gamma \in [1, 5/3)$),

$$\sum_{s_n \leq x} (s_{n+1} - s_n)^{\gamma} \asymp x(\log x)^{\frac{1}{2}(\gamma - 1)}$$

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- That is to say, for any $h \geq 1$, the number of $x \in [X, 2X]$ for which

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- Again this is based on the shifted convolution problem for $r_K(n)$ and does not extend beyond quadratic number fields.
- If, like Hooley and Plaksin, we do not request asymptotic formula, we get an improved bound for our exceptional set for any K.

Gaps between norm forms

Theorem (M-Radziwiłł (202?))

Let K be a number field and let $\delta_K(x)$ be the density of norm-forms. Then, for any $\varepsilon > 0$, there exists a constant $c = c(K, \varepsilon)$ such that, for any $h \geq 1$, one has

$$\frac{1}{h\delta_K(x)^{-1}} \sum_{x < n \leq x + h\delta_K(x)^{-1}} g_K(n) \geq c\delta_K(x)$$

for all but $O_{\varepsilon,K}(Xh^{-1/2+\varepsilon})$ of $x \in (X, 2X]$.

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$$\frac{1}{h\delta_K(x)^{-1}} \sum_{x < n \leq x + h\delta_K(x)^{-1}} g_K(n) \geq c\delta_K(x)$$

for all but $O_{\varepsilon,K}(Xh^{-1/2+\varepsilon})$ of $x \in (X, 2X]$. Consequently, letting $1 \leq n_1 < n_2 < \ldots$ denote the sequence of positive norm-forms of K, one has for any $\gamma \in [1, 3/2)$,

$$\sum_{n_i \leq x} (n_{i+1} - n_i)^\gamma \asymp_{\gamma,K} x\delta_K(x)^{\gamma-1}.$$

Gaps between norm forms

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This vastly extends Hooley's and Plaksin's results (case $K = \mathbb{Q}(i)$ with $\gamma \in [1, 5/3)$ and $\gamma \in [1, 2)$ repsectively).

Other multiplicative functions

Also these results work more generally. E.g. we get

Corollary

Let $\varepsilon > 0$ be given and $h \geq 1$. Then the number of intervals $[x, x+h]$ with $x \in [X, 2X]$ that do not contain an x^ε-smooth number is $\ll_{\eta,\varepsilon} Xh^{-1/2+\eta}$ for all $\eta > 0$.
Consequently, letting $1 \leq n_1 < n_2 < \dots$ denote the sequence of integers n such that all prime factors of n are $\leq n^\varepsilon$, one has, for any $\gamma \in [1, 3/2)$,

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This improves on a recent result of Heath-Brown who got for (3) the upper bound $\ll x^{1+\eta}$ for any $\eta > 0$.

Recalling the main theorem

Theorem (M-Radziwiłł (202?))

Let $\varepsilon > 0$. *Let* $f : \mathbb{N} \to [-1, 1]$ *be a multiplicative function s.t.*

$$\sum_{w < p \le z} \frac{|f(p)|}{p} \ge \varepsilon \sum_{w < p \le z} \frac{1}{p} + O\left(\frac{1}{\log w}\right)$$

for all $2 \le w < z < x^{\varepsilon}$. *Set* $h_1 := \prod_{p \le x}\left(1 + \frac{1-|f(p)|}{p}\right)$.

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for all $2 \leq w < z < x^{\varepsilon}$. *Set* $h_1 := \prod_{p\leq X}\left(1 + \frac{1-|f(p)|}{p}\right)$. *For any* $\delta > 0$, *and any* $h_0 \geq 1$,

$$\left|\frac{1}{h_0 h_1} \sum_{x<n\leq x+h_0 h_1} f(n) - \frac{1}{X} \sum_{X<n\leq 2X} f(n)\right| \leq \delta \prod_{p\leq X}\left(1 + \frac{|f(p)|-1}{p}\right)$$

for all but at most $O(Xh_0^{-c\delta^{\kappa}})$ *integers* $x \in (X, 2X]$, *for some* $c = c(\varepsilon)$ *and* $\kappa = \kappa(\varepsilon) > 0$.

Proof ideas

- For simplicity concentrate on case when the average of f is 0.
- Our starting point is Perron's formula, giving

$$\sum_{x<n\leq x+H} f(n) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \sum_{X<n\leq 3X} \frac{f(n)}{n^{1+it}} \cdot \frac{(x+H)^{1+it} - x^{1+it}}{1+it}$$

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- Normally one would go on and average over x, getting

$$\frac{1}{X}\int_X^{2X} \left| \sum_{x<n\leq x+H} f(n) \right|^2 dx \ll \int_{-X/H}^{X/H} \left| \sum_{X<n\leq 3X} \frac{f(n)}{n^{1+it}} \right|^2 dt.$$

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- Recall $H = h_0 h_1$ with $h_1 = \prod_{p\leq X}(1+\frac{1-|f(p)|}{p})$. Now to show that

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with $O(Xh_0^{-c\delta^\kappa})$ exceptions, we would need the bound

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Studying the mean square

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- Using Shiu's and Henriot's bounds for averages and correlations of multiplicative functions, one can tweak the usual MVT to show that, for any $a_n \leq |f(n)|$,

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- After reproving Halasz and Lipschitz type estimates in the sparse setting, we can repeat those arguments.
- But this gives about $h_0^{-c\delta^\kappa} + (\log X)^{-\kappa}$ where we want $h_0^{-c\delta^\kappa}$.

An issue with mean square

- Actually showing the bound

$$\int_{-X/H}^{X/H} \left| \sum_{X<n\leq 3X} \frac{f(n)}{n^{1+it}} \right|^2 dt \ll \delta^2 h_0^{-c\delta^\kappa} \prod_{p\leq X} \left(1 + \frac{|f(p)|-1}{p}\right)^2.$$

in general is not possible — there might be some points t where $\sum f(n) n^{-1+it}$ is $\asymp \prod_{p\leq X}(1+\frac{|f(p)|-1}{p})(\log X)^{-\kappa}$.

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$$\sum_{P<p\leq 2P} \frac{f(p)}{p^{1+it}} = O(P^{-1/4+\varepsilon})$$

for all $P = 2^j \in [X^{\varepsilon^3}, X^{\varepsilon^2}]$, then that method would give the desired bound. (also we need to construct a good sieve majorant for $f(n)$ to handle "$n \notin \mathcal{S}$")

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- Key new idea: Handle "exceptional" t before taking the mean square over x.

Splitting into $\mathcal{T}$ and $\mathcal{U}$

- Recall

$$\sum_{x<n\leq x+H} f(n) \approx \frac{H}{2\pi i}\int_{-X/H}^{X/H} \sum_{X<n\leq 3X} \frac{f(n)}{n^{1+it}} x^{it}\, dt.$$

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- Split $[-X/H, X/H] = \mathcal{T} \cup \mathcal{U}$ with $t \in \mathcal{T}$ iff

$$\left| \sum_{P<p\leq 2P} \frac{f(p)}{p^{1+it}} \right| < P^{-1/4+\varepsilon} \quad \text{for all } P = 2^j \in [X^{\varepsilon^3}, X^{\varepsilon^2}].$$

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- By MVT $|\mathcal{U}| \leq (X/H)^{1/2-\varepsilon}$, and by previous discussion,

$$\frac{1}{X}\int_X^{2X} \left| \frac{H}{2\pi i}\int_{\mathcal{T}} \sum_{X<n\leq 3X} \frac{f(n)}{n^{1+it}} x^{it}\, dt \right|^2 dx$$

$$\ll \int_{\mathcal{T}} \left| \sum_{X<n\leq 3X} \frac{f(n)}{n^{1+it}} \right|^2 dt \ll \delta^2 h_0^{-c\delta^\kappa} \prod_{p\leq X} \left(1 + \frac{|f(p)|-1}{p}\right)^2.$$

Handling $\mathcal{U}$

- We are left with studying, for certain $|\mathcal{U}| \leq (X/H)^{1/2-\varepsilon}$,

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- Since most integers have at least two prime factors from $(X^{\varepsilon^2}, X^{\varepsilon}]$, we can at least morally replace $\sum \frac{f(n)}{n^{1+it}}$ by

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- Now, by Huxley's large value theorem, those $t \in \mathcal{U}$ for which

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give an acceptable contribution to square mean of (4).

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give an acceptable contribution to square mean of (4).
- The complement is tiny and there (4) is $o(H)$ by Halász + tailored Halász-Montgomery type large value results.

The results with positive proportion lower bound

- When one only wants, for $f: \mathbb{N} \to [0,1]$, with a good exceptional set,

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$$\frac{1}{H} \sum_{x<n<x+H} f(n) \geq \delta \prod_{p \leq x} \left(1 + \frac{f(p)-1}{p}\right),$$

it suffices to show that, for $K = \lfloor 1/\varepsilon^{10} \rfloor$,

$$\frac{1}{H} \sum_{\substack{x<p_1\cdots p_{K-1}m \leq x+H \\ p_j \in [X^{(1-\varepsilon^{10})/K}, X^{(1+\varepsilon^{10})/K}]}} f(p_1)\cdots f(p_{K-1}) f(m)$$

$$\gg \prod_{p \leq x} \left(1 + \frac{f(p)-1}{p}\right).$$

- The resulting Dirichlet polynomial is a product of short factors. This gives a lot more flexibility with applying mean and large value theorems
- This way we get the desired result.

Thank you!