Extremely Pointless Curves

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Work

This is ongoing joint work with Xander Faber.

Gonality

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- Gonality 1 curves are isomorphic to $\mathbb{P}^1$, so coincide with genus 0 curves.
- Gonality 2 curves are **hyperelliptic**, and include elliptic curves (genus 1 and up).
- Gonality 3 curves are known as **trigonal** curves.

Gonality, Genus, and Curves over Finite Fields

- A natural question (indeed, one asked by Van der Geer) is, given a smooth, projective curve over a finite field $\mathbb{F}_q$ of genus g and gonality γ, what is the maximum number of points?
- We answered this for $q \leq 4$ and $g \leq 5$ in previous work.
- We used a combination of explicit geometry of small-genus curves, as well as computer searches.

Enter Pointless Curves

- Let C be a curve with genus $g > 0$ over a finite field
- The gonality satisfies $\gamma \leq g + 1$.
- If C has a rational point, then the gonality satisfies $\gamma \leq g$.
- A curve with gonality $g + 1$ must thus be **pointless**, a concept introduced by Howe-Lauter-Top.
- In fact, a curve over $\mathbb{F}_q$ with gonality $g + 1$ has no effective divisor of degree $g - 2$.
- That implies it has no points over $\mathbb{F}_{q^r}$ for all $r | g - 2$.
- Since such a curve is pointless over a number of different finite fields, we call it **extremely pointless**.

Weil Bounds

- Applying Weil's Formula to a pointless curve over $\mathbb{F}_{q^{g-2}}$, we have $2g\sqrt{q^{g-2}} \geq q^{g-2} + 1 > q^{g-2}$.
- Thus $q < (2g)^{\frac{2}{g-2}}$.
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- Thus $q < (2g)^{\frac{2}{g-2}}$.
- This bound gives us the following list of possibilites for extremely pointless curves.
- $g = 3$ and $q \leq 32$;
- $g = 4$ and $q \leq 7$;
- $g = 5$ and $q \leq 4$;
- $g = 6$ and $q = 2$ or 3; or
- $7 \leq g \leq 10$ and $q = 2$.

Previous Results for Genus 3, 4 and 5

- There exists an extremely pointless curve of genus 3 over $\mathbb{F}_q$ if and only if $q \leq 23$ or $q = 29$ or $q = 32$ (Howe-Lauter-Top).
- There exists an extremely pointless curve of genus 4 over $\mathbb{F}_2$. (Faber-G.)
- There exists an extremely pointless curve of genus 4 over $\mathbb{F}_3$. (Castryck-Tuitman)
- There does not exist an extremely pointless curve of genus 4 over $\mathbb{F}_4$. (Faber-G.)
- There does not exist an extremely pointless curve of genus 5 over $\mathbb{F}_2$, $\mathbb{F}_3$ or $\mathbb{F}_4$. (Faber-G.)

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- There does not exist an extremely pointless curve of genus 5 over $\mathbb{F}_2$, $\mathbb{F}_3$ or $\mathbb{F}_4$. (Faber-G.)
- (Our treatment of binary curves is on the arXiv; our treatment of ternary and quaternary curves will be soon.)

What's Left

- Eight cases:
- $g = 4$ and $q = 5$ or $q = 7$;
- $g = 6$ and $q = 2$ or 3; or
- $7 \leq g \leq 10$ and $q = 2$.

Lauter's Algorithm for Serre's Explicit Method

- In 1998, Lauter gave an algorithmic description of Serre's technique that computes a list of all possible zeta functions of a curve over a finite field.
- For an extremely pointless curve, certain terms must be zero, hence we can eliminate most zeta functions.
- For the (g, q) pairs $(4, 5)$, $(4, 7)$, $(6, 3)$ and $(8, 2)$, $(10, 2)$ a computation using Lauter's algorithm eliminates all zeta functions.

The Stubborn ~~Three~~ Two

- There exists an extremely pointless curve of genus 3 if and only if $q \leq 23$ or $q = 29$ or $q = 32$.
- There exists an extremely pointless curve of genus 4 if and only if $q = 2$ or 3.
- We don't know if there is an extremely pointless curve of genus g over $\mathbb{F}_q$ for these (g, q)-pairs:
  - ~~$(6, 2)$ — 3 zeta functions survive!~~
  - $(7, 2)$ — 79 zeta functions survive!
  - $(9, 2)$ — 1 zeta function survives!
  - Further tools by Serre and Howe-Lauter gets us down to $\{\not{2},77, 1\}$ survivors.
- We can exclude all other cases.

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- Questions welcome; answers all the more so.