On the automorphic Langlands group

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Outline of the talk.

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- Universal mapping property of free products
- Definition of restricted free products of topological groups
- Universal mapping property of restricted free products
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Automorphic Langlands group L_K of a number field K
- Notation
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- Weil-Arthur idèles of K
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Notation.

- $\{G_i\}_{i \in I}$: a collection of k_ω-topological groups, where the index set I is countable.
- For all but finitely many $i \in I$, let O_i be a fixed open subgroup of G_i.
- I_∞: the finite subset of I consisting of all $i \in I$ for which O_i is **not** defined.

References.

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Universal mapping property of free products.

- Let $*_{i \in I} G_i$ denote the free product of the collection $\{G_i\}_{i \in I}$ together with the canonical embeddings

$$\iota_{i_o} : G_{i_o} \hookrightarrow \underset{i \in I}{*} G_i,$$

  for each $i_o \in I$.
- **The universal mapping property of free products:** Let H be a topological group s.t. $\forall i_o \in I$, $\exists$ a cont. homomorphism $\phi_{i_o} : G_{i_o} \to H$.
  THEN: $\exists!$ cont. homomorphism $\phi : *_{i \in I} G_i \to H$, such that $\phi \circ \iota_{i_o} = \phi_{i_o}$, for every $i_o \in I$.

Definition (Restricted free products of top. groups).

- For every finite subset S of I satisfying $I_\infty \subseteq S$, define the topological group

$$G_S := \mathop{*}_{i \notin S} O_i * \left(\mathop{*}_{i \in S} G_i \right)$$

as the free product of the topological groups O_i, for $i \in I - S$, and G_i, for $i \in S$.

- G_S exists in the category of topological groups.
- For finite subsets S and T of I, such that $I_\infty \subseteq S \subseteq T$, the continuous homomorphism

$$\tau_S^T : G_S \to G_T$$

for $S \subseteq T$ is defined naturally by the *"universal mapping property of free products"*.

- The *restricted free product of the collection* $\{G_i\}_{i\in I}$ *with respect to the collection* $\{O_i\}_{i\in I-I_\infty}$, which is denoted by $*'_{i\in I}(G_i : O_i)$, is defined by the injective limit

$$\mathop{*}_{i\in I}{}'(G_i : O_i) := \varinjlim_S G_S$$

of G_S over all possible such finite $S \subset I$ s.t. $I_\infty \subseteq S$, where the connecting morphism are

$$\tau_S^T : G_S \to G_T$$

for $S \subseteq T$.

- The topology on $*'_{i\in I}(G_i : O_i)$: defined by declaring $X \subseteq *'_{i\in I}(G_i : O_i)$ to be open if $X \cap G_S$ is open in G_S for every S. So, endowed with this topology, $*'_{i\in I}(G_i : O_i)$ is a topological group. **This is the place where the assumption that I is countable and $\forall i \in I$, G_i is a k_ω-group is used.**

Universal mapping property of restricted free products.

- Let H be a topological group.
- Assume: $\forall i \in I$, $\exists$ a cont. homomorphism

$$\phi_i : G_i \to H.$$

THEN,

- $\exists!$ cont. homomorphism $\phi_S : G_S \to H$, $\forall$ finite $S \underset{finite}{\subset} I$ s.t. $I_\infty \subseteq S$, and
- $\exists!$ cont. homomorphism $\phi = \varinjlim_{S} \phi_S : *'_{i \in I}(G_i : O_i) \to H$

satisfying

$$\phi_S = \phi \circ c_S : G_S \xrightarrow{c_S} *'_{i \in I}(G_i : O_i) \xrightarrow{\phi} H,$$

where $c_S : G_S \to *'_{i \in I}(G_i : O_i)$ is the canonical hom., $\forall S$.

Why restricted free products ?

- Because :

$$*'_{i\in I}(G_i : O_i) \xrightarrow{\mathrm{ab}} (*'_{i\in I}(G_i : O_i))^{\mathrm{ab}} \xrightarrow{\sim} \prod_{i\in I}{}'(G_i^{ab} : O_i^{ab}).$$

  Here, $\prod'_{i\in I}(G_i^{ab} : O_i^{ab})$ is the restricted direct product of the collection $\{G_i^{\mathrm{ab}}\}_{i\in I}$ w.r.t. the collection $\{O_i^{\mathrm{ab}}\}_{i\in I-I_\infty}$.
- Choosing the index set I as the set of places of a global field K, the groups G_i for $i \in I$, and O_i for $i \in I - I_\infty$ as certain "arithmetical objects attached to the global field K" in such a way that $G_i^{\mathrm{ab}} \simeq K_i^\times$ and $O_i^{\mathrm{ab}} \simeq U_{K_i}$ for places i of K, this group may be viewed as **a non-commutative generalization of $\mathbb{J}_K$, the idèle group of K.**
- Such a non-abelian generalization of the idèle group $\mathbb{J}_K$ of K is **only possible, if we have a reasonable local non-abelian class field theory over K_ν in the sense of Hasse, for finite places ν of K.**

Notation.

- $K :=$ a number field (or more generally a global field).
- $\mathbf{h}_K = \mathbf{f}_K :=$ the set of all finite places of K.
- $\mathbf{a}_K = \infty_K :=$ the set of all infinite places of K.
- $K_\nu :=$ the ν-adic completion of K at a place ν of K.

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Local non-abelian reciprocity map of K_ν ($\nu \in \mathbf{h}_K$)

The groups $\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})}$ for $\upsilon \in \mathbf{h}_K$

The aim here is to review very briefly the references [5,8] .

- For $\nu \in \mathbf{f}_K$, we fix a lifting (=a Lubin-Tate splitting) φ_{K_ν} of the Frobenius automorphism Frob_{K_ν} of K_ν^{nr} to K_ν^{sep}.
- There exists a topological group $\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})}$ depending on K_ν, **whose construction uses the theory of APF-extensions and fields of norms of Fontaine-Wintenberger.**
- The topological group $\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})}$ comes equipped with a topological isomorphism

$$\{\bullet, K_\nu\}_{\varphi_\nu}^{\mathrm{Galois}} : \nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})} \xrightarrow{\sim} G_{K_\upsilon},$$

  we call **the local non-abelian norm residue isomorphism of** K_ν, because it very much behaves like local abelian norm residue map of K_ν.
- In what follows, we shall consider the "Weil form" of the local non-abelian norm residue isomorphism

$$\{\bullet, K_\nu\}_{\varphi_\nu}^{\mathrm{Weil}} : {}_{\mathbb{Z}}\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})} \xrightarrow{\sim} W_{K_\upsilon},$$

  of K_υ.

Local non-abelian reciprocity map of K_ν ($\nu \in \mathbf{h}_K$)

Ramification filtration on W_{K_ν} in upper numbering

- There exists a subgroup ${}_{\mathbb{Z}}\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})\underline{o}}$ of ${}_{\mathbb{Z}}\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})}$ so that the "Weil form" of the local non-abelian norm residue isomorphism $\{\bullet, K_\nu\}_{\varphi_\nu}^{\mathrm{Weil}}$ of K_υ induces an isomorphism

$$\{\bullet, K_\nu\}_{\varphi_\nu}^{\mathrm{Weil}} : {}_{\mathbb{Z}}\nabla_{K_\upsilon}^{(\varphi_{K_\upsilon})\underline{o}} \xrightarrow{\sim} W_{K_\upsilon}^{\underline{o}},$$

of topological groups (for details look at [6]).

The well-known "local abelian class field theory" and the "local non-abelian class field theory" can be summarized and associated via the following tables :

Local non-abelian reciprocity map of K_ν $(\nu \in \mathbf{h}_K)$

Summary

Non-abelian local C.F.T. (φ_K fixed)	
G_{K_ν}	$\nabla_{K_\nu}^{(\varphi_{K_\nu})}$
W_{K_ν}	${}_{\mathbb{Z}}\nabla_{K_\nu}^{(\varphi_{K_\nu})}$
$W_{K_\nu}^0$	${}_1\nabla_{K_\nu}^{(\varphi_{K_\nu})^{\underline{0}}}$
$W_{K_\nu}^\delta,\ \delta \in (i-1, i]$	${}_1\nabla_{K_\nu}^{(\varphi_{K_\nu})^{\underline{i}}}$

and **via abelianization**:

Abelian local class field theory	
$G_{K_\nu}^{ab}$	$\widehat{K_\nu^\times}$
$W_{K_\nu}^{ab}$	$K_\nu^\times$
${W_{K_\nu}^{ab}}^0$	U_{K_ν}
${W_{K_\nu}^{ab}}^\delta,\ \delta \in (i-1, i]$	$U_{K_\nu}^i$

The local Langlands group L_{K_ν} of K_ν ($\nu \in \mathbf{h}_K \cup \mathbf{a}_K$)

- The absolute Langlands group L_{K_ν} of K_ν (which exists!) is defined by:
  - $L_{K_\nu} := WA_{K_\nu} := W_{K_\nu} \times \mathrm{SU}(2,\mathbb{R})$, if $\nu \in \mathbf{h}_K$;
  - $L_{K_\nu} := W_{K_\nu}$, if $\nu \in \mathbf{a}_K$,

  where W_{K_ν} denotes the Weil group of K_ν. Recall: $W_{\mathbb{C}} = \mathbb{C}^\times$ and $W_{\mathbb{R}} = \mathbb{C}^\times \cup j\mathbb{C}^\times$.
- For $\nu \in \mathbf{h}_K$, fix a Lubin-Tate splitting φ_{K_ν}. The local non-abelian norm residue isomorphism

$$\{\bullet, K_\nu\}_{\varphi_\nu}^{\mathrm{Weil}} : {}_{\mathbb{Z}}\nabla_{K_\nu}^{(\varphi_{K_\nu})} \xrightarrow{\sim} W_{K_\nu}$$

  of K_ν in "Weil form" induces an isomorphism

$$\{\bullet, K_\nu\}_{\varphi_\nu}^{\mathrm{Langlands}} : {}_{\mathbb{Z}}\nabla_{K_\nu}^{(\varphi_{K_\nu})} \times \mathrm{SU}(2,\mathbb{R}) \xrightarrow[\sim]{\{\bullet, K_\nu\}_{\varphi_{K_\nu}}^{\mathrm{Weil}} \times \mathrm{id}_{\mathrm{SU}(2,\mathbb{R})}} L_{K_\nu},$$

  the local non-abelian norm residue isomorphism of K_ν in "Langlands form".

Weil-Arthur idèles of K

Fix $\underline{\varphi} = \{\varphi_{K_\nu}\}_{\nu \in \mathbf{h}_K}$.

- Define an **unconditional** non-commutative topological group $\mathcal{WA}_K^{\underline{\varphi}}$ depending only to the number field K, which we called **the Weil-Arthur idèle group of** K, by the restricted free product

$$\mathcal{WA}_K^{\underline{\varphi}} := \underset{\nu \in \mathbf{h}_K}{{\ast}'} \left({}_{\mathbb{Z}}\nabla_{K_\nu}^{(\varphi_{K_\nu})} \times \mathrm{SU}(2,\mathbb{R}) : {}_{1}\nabla_{K_\nu}^{(\varphi_{K_\nu})^{0}} \times \mathrm{SU}(2,\mathbb{R}) \right) * W_{\mathbb{R}}^{*r_1} * W_{\mathbb{C}}^{*r_2}$$

  of the collection $\{{}_{\mathbb{Z}}\nabla_{K_\nu}^{(\varphi_{K_\nu})} \times \mathrm{SU}(2,\mathbb{R})\}_{\nu \in \mathbf{h}_K} \cup \{W_{K_\nu}\}_{\nu \in \mathbf{a}_K}$ with respect to the collection $\{{}_{1}\nabla_{K_\nu}^{(\varphi_{K_\nu})^{0}} \times \mathrm{SU}(2,\mathbb{R})\}_{\nu \in \mathbf{h}_K}$. Here, $r_1 = \#(\mathbf{a}_{K,\mathbb{R}})$ and $2r_2 = \#(\mathbf{a}_{K,\mathbb{C}})$.
- The topological group $\mathcal{WA}_K^{\underline{\varphi}}$ can be considered as a non-commutative generalization of the idèle group $\mathbb{J}_K$ of K, because ${\mathcal{WA}_K^{\underline{\varphi}}}^{ab} = \mathbb{J}_K$.

Automorphic Langlands group L_K of K

- Let L_K denote the hypothetical automorphic Langlands group L_K of the number field K.
  Assumption: Assume that L_K exists for now.
- It is expected that, an embedding $e_\nu : K^{sep} \hookrightarrow K_\nu^{sep}$ determines a homomorphism (unique up to L_K-conjugacy) $e_\nu^{\mathrm{Langlands}} : L_{K_\nu} \to L_K$.
- Therefore, for $\nu \in \mathbf{h}_K$, there exists a morphism

$$_{\mathbb{Z}}\nabla_{K_\nu}^{(\varphi_{K_\nu})} \times \mathrm{SU}(2,\mathbb{R}) \xrightarrow[\sim]{\{\bullet, K_\nu\}_{\varphi_{K_\nu}}^{\mathrm{Langlands}}} L_{K_\nu} \xrightarrow{e_\nu^{\mathrm{Langlands}}} L_K$$

  (unique up to L_K-conjugacy).

So, by the universal mapping property of restricted free products, we state **the main result of our talk:**

Theorem (The global non-abelian norm residue map of K in "Langlands form")

The collection of arrows $\{e_{\nu}^{\mathrm{Langlands}} \circ \{\bullet, K_{\nu}\}_{\varphi_{K_{\nu}}}^{\mathrm{Langlands}}\}_{\nu \in \mathbf{h}_K}$ *defines a unique continuous homomorphism*

$$\mathsf{NR}_{K}^{\varphi^{\mathrm{Langlands}}} : \mathcal{WA}_{K}^{\varphi} \to L_K,$$

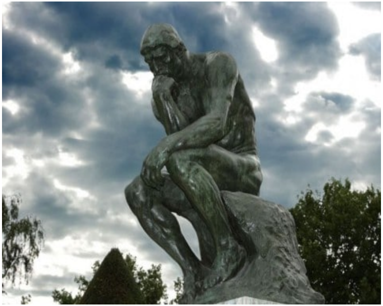
which is unique up to "local L_K-conjugation".

- Moreover, this result is compatible with Arthur's construction of L_K (look at [2]).
- The arrow $\mathsf{NR}_{K}^{\varphi^{\mathrm{Langlands}}} : \mathcal{WA}_{K}^{\varphi} \to L_K$ behaves like global abelian norm residue map of K (look at [3]).

We conclude our talk with the following conjecture:

Conjecture

The homomorphism $\mathsf{NR}_{K}^{\varphi^{\mathrm{Langlands}}} : \mathcal{WA}_{K}^{\varphi} \to L_K$ *is open and surjective.*



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