The non-vanishing spectrum of arithmetic progressions of squares

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Arithmetic Progressions of Squares

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We also know (due to Euler[1], among others) that there are not any such nontrivial progressions longer than length-three. So henceforth we are going to just drop the term *length-three* as being redundant, and we will further abbreviate arithmetic progressions as APs.

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We have devised a way of counting the number of APs of primitive integer squares given certain restrictions to the size of the integers.

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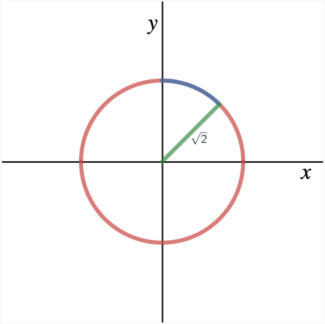

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Theorem 1 (H., Kuan, Lowry-Duda, Walker, 2020)[2]

Fix $\delta \in [0, 1]$. For any $\epsilon > 0$, the number of primitive APs of squares $\{a^2, b^2, c^2\}$ with $b^2 \leq X$ and $(a/b)^2 \leq \delta$ is

$$\frac{2}{\pi^2} \arcsin(\sqrt{\delta/2}) X^{\frac{1}{2}} + O_\epsilon(X^{\frac{3}{8}+\epsilon}).$$

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and observe it can also be stated as an equidistribution result:

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For any $\epsilon > 0$, the number of reduced rational points $\left(\frac{a}{b}, \frac{c}{b}\right)$ on a circle with radius $\sqrt{2}$ with $b \le X$ within a sector of angle ω is

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The main term of this asymptotic is not difficult to see using elementary methods[5], but the error term is nontrivial.

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Theorem 2 (H., Kuan, Lowry-Duda, Walker, 2020)[2]

The number of APs with $c^2 \leq X$ is

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Theorem 3 (H., Kuan, Lowry-Duda, Walker, 2020)[2]

Suppose that $Y \leq X$. The number of APs with $a^2 \leq Y$ and $b^2 \leq X$ is

$$\frac{1}{\sqrt{2}\pi^2} Y^{\frac{1}{2}} \log\big(X/Y\big) + \frac{\sqrt{2}\log(e(4-2\sqrt{2}))}{\pi^2} Y^{\frac{1}{2}} + O_\epsilon\big(X^{\epsilon} Y^{\frac{3}{8}+\epsilon}\big).$$

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Theorem 4 (H., Kuan, Lowry-Duda, Walker, 2020)[2]

The number of primitive APs with $ab \leq X$ is

$$\frac{2\sqrt{2}}{\pi^2} {}_2F_1(\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{5}{4}, \tfrac{1}{2}) X^{\frac{1}{2}} + O_\epsilon\left(X^{\frac{3}{8}+\epsilon}\right).$$

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$$\mathcal{D}(s,w) := \sum_{\substack{m,n=1 \\ (m,n)=1}}^{\infty} \frac{r_1(h)r_1(m)r_1(2m-h)}{m^s h^w}$$

where $r_1(n)$ is the number of ways n can be written as the square of an integer. So the coefficients of each summand essentially determine whether or not $\{h, m, 2m-h\}$ is an arithmetic progression of primitive squares since

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In particular, we are able to derive a meromorphic continuation of $\mathcal{D}(s,w)$ to all $(s,w) \in \mathbb{C}^2$ by means of a spectral expansion. Once we have a thorough understanding of the analytic behavior of the above series, we can obtain our aforementioned asymptotic results by carefully taking inverse Mellin transforms.

To do this, we will take advantage of the automorphic properties of theta functions.

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For $N \in \mathbb{N}$, let $\Gamma_0(N)$ denote the congruence subgroup:

$$\Gamma_0(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; N|C \right\}.$$

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It is easy to show that $\Gamma_0(N)$ acts on $\mathbb{H}$ by Möbius Maps:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} z = \frac{Az + B}{Cz + D}.$$

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For $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0(4)$, applying Poisson's summation formula on the generators of $\Gamma_0(4)$ allows us to prove that

$$\theta(\gamma z) = \left(\tfrac{C}{D}\right) \epsilon_D^{-1} \sqrt{Cz + D}\, \theta(z),$$

where $\left(\frac{C}{D}\right)$ denotes Shimura's extension of the Jacobi symbol and $\epsilon_D = 1$ or i depending on if $D \equiv 1$ or $3 \pmod 4$, respectively.[4]

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It turns out that $\theta(2z)$ is also a holomorphic form of $\Gamma_0(8)$ with nebentypus $\chi(d) := \left(\frac{2}{d}\right)$.

Thus have that $V(z) := y^{\frac{1}{2}}\theta(2z)\overline{\theta(z)}$ is a weightless automorphic function on $\Gamma_0(8)$ with nebentypus χ, that is

$$V(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z) = \chi(d)V(z) \quad \text{for} \quad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(8).$$

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While $V(z)$ is an automorphic function of $\Gamma_0(8)$ and character χ, it is not square-integrable.

Let

$$P_h(z, s; \chi) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(8)} \chi(\gamma) \Im(\gamma z)^s e(h\gamma z)$$

denote the level 8, twisted Poincaré series.

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$$\sum_{h=1}^{\infty} \frac{\langle V, P_h(\cdot,\overline{s}+\frac{1}{2};\chi)\rangle}{h^w} = \frac{\Gamma(s)}{(8\pi)^s} \sum_{m=1}^{\infty} \frac{r_1(h)r_1(m)r_1(2m-h)}{m^s h^w}.$$

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From there we *wish* to take a spectral expansion of $P_h(\cdot,\bar{s};\chi)$ and rewrite the left-hand side of the above equation as a sum of eigenfunctions and so obtain a meromorphic continuation of the above shifted Dirichlet series.

However we require $V(z)$ to be in $\mathcal{L}^2(\Gamma_0(8),\chi)$ to guarantee this spectral expansion. Thus we have to regularize $V(z)$.

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Let $E(z,s;\chi)$ denote the weight 0, level 8 Eisenstein series with character $\chi := \left(\frac{2}{d}\right)$,

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It turns out that $E(z,\frac{1}{2};\chi)$ also only has polynomial growth at ∞ and 0, and it matches that of $y^{\frac{1}{2}}\theta(2z)\overline{\theta(z)}$ at each cusp. What remains has exponential decay and so we have that:

$$\widetilde{V}(z) := y^{\frac{1}{2}}\theta(2z)\overline{\theta(z)} - E(z,\tfrac{1}{2};\chi) \in \mathcal{L}^2(\Gamma_0(8),\chi).$$

Since $\widetilde{V}(z) \in \mathcal{L}^2(\Gamma_0(8),\chi)$, it has a spectral decomposition.

The Non-Vanishing Spectrum

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$$f(z) = \sum_j \langle f, \mu_j \rangle \mu_j(z) + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{\mathbb{R}} \langle f, E_{\mathfrak{a}}(\cdot, \tfrac{1}{2} + it; \chi) \rangle E_{\mathfrak{a}}(z, \tfrac{1}{2} + it; \chi)\, dt,$$

as summarized by Michel[3]. Here $\{\mu_j\}$ denotes an orthonormal basis of Maass cusp forms in $\mathcal{L}^2(\Gamma_0(8), \chi)$, the **discrete spectrum**, and $E_{\mathfrak{a}}(s, z; \chi)$ is the Eisenstein series for level $\Gamma_0(8)$ with character χ for the singular cusp $\mathfrak{a}$, which correspond to the **continuous spectrum**.

Since Eisenstein series on $\Gamma_0(8)$ with $\chi(d) = \left(\frac{2}{d}\right)$ only have two singular cusps, 0 and ∞, the continuous spectrum only has summands arising from those cusps. Furthermore $\langle \widetilde{V}(z), E_{\mathfrak{a}}(\cdot, \frac{1}{2} + it; \chi)\rangle = 0$ for both cusps since the constant term of the Fourier expansion of $\widetilde{V}(z)$ is zero at both cusps.

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Furthermore, $\langle E(z, \frac{1}{2}; \chi), \mu_j\rangle = 0$ for all μ_j and so the spectral expansion simplifies to

$$\widetilde{V}(z) = \sum_{j \neq 0} \langle V, \mu_j\rangle \mu_j(z)$$

where we recall that $V(z) = y^{\frac{1}{2}} \theta(2z)\overline{\theta(z)}$.

When we computed this, we originally thought we had stumbled into a contradiction, since we thought $\langle V, \mu_j \rangle$ should be zero for all Maass forms (see last year's Maine-Québec Number Theory Conference Talk *The Impossible Vanishing Spectrum*).

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The main reason for this confusion, among other things, was the observation that $\langle V, \mu_j \rangle$ has as a factor:

$$\operatorname*{Res}_{s=1} L(s, \mathrm{Sym}^2 \mu_j),$$

and we believed the symmetric square L-functions of Maass forms were always entire, which would mean the above residue would be zero.

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That is to say Maass forms whose L-functions are also the L-functions of Hecke characters. The Dihedral Maass forms are precisely characterized by the above non-vanishing criteria, and what's more, the Dihedral Maass forms for $\mathcal{L}^2(\Gamma_0(8), \chi)$ are relatively easy (compared to non-Dihedral Maass forms) to explicitly characterize.

So it turns out we get a very explicit spectral expansion for

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Theorem (H., Kuan, Lowry-Duda, Walker, 2020[2])

The double Dirichlet series $\mathcal{D}(s,w)$ has meromorphic continuation to $\mathbb{C}^2$. For $\operatorname{Re} s$ and $\operatorname{Re} w$ sufficiently large, we have

$$\mathcal{D}(s,w) = \frac{2^{3s}(1-2^{-2s-2w})}{\zeta^{(2)}(4s+4w)\log(1+\sqrt{2})\Gamma(2s)} \times \sum_{m\in\mathbb{Z}} (-1)^m L(2s+2w, \eta^{2m}) \Gamma(s+it_m)\Gamma(s-it_m)$$

in which $t_m = \frac{m\pi}{2\log(1+\sqrt{2})}$, $\zeta^{(2)}(s) = (1-\frac{1}{2^s})\zeta(s)$,and η is the Hecke character defined on ideals of $\mathbb{Q}(\sqrt{2})$ by

$$\eta\big((a+b\sqrt{2})\big) = \operatorname{sgn}(a+b\sqrt{2})\operatorname{sgn}(a-b\sqrt{2})\left|\frac{a+b\sqrt{2}}{a-b\sqrt{2}}\right|^{\frac{i\pi}{2\log(1+\sqrt{2})}}.$$

Theorems 1-4, given at the beginning of this talk were all obtained using the Phragmén-Lindelöf convexity bound for $L(s, \eta^{2m})$ in vertical strips.

$$L(s, \eta^{2m}) \ll (1 + |s + it_m|)^{\frac{1}{4}+\epsilon}(1 + |s - it_m|)^{\frac{1}{4}+\epsilon}$$

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But we can obtain slight improvements on our error terms by using known subconvexity bounds for $L(s, \eta^{2m})$:

$$L(s, \eta^{2m}) \ll (1 + |s + it_m|)^{\alpha}(1 + |s - it_m|)^{\alpha}.$$

where $\alpha \leq \frac{1}{4}$.

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In particular, all of the $\frac{3}{8} + \epsilon$ exponents in the errors terms of Theorems 1-4 can be replaced with $\frac{1}{2} - \frac{1}{6+8\alpha} + \epsilon$.

The current best-known progress for α [6] is $\alpha \leq \frac{103}{512}$, which would yield an exponent of $\frac{359}{974} + \varepsilon \leq 0.36859 + \varepsilon$. Under the Lindelöf Hypothesis, we can assume $\alpha = 0$ which would yield and exponent of $\frac{1}{3} + \varepsilon$.

Notably, the techniques for obtaining the spectral expansion of $\mathcal{D}(s, w)$ could be generally applied to obtain the spectral expansion of any sum of the form:

$$\sum_{m \geq 1} \frac{r_1(m) r_1(tm \pm h)}{m^s}.$$

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When $t = 5$ and $h = 4$, the series

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We are presently exploring how sums of the above type may be used to characterize other families of second order linear recurrence relations.

Thanks!

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