Formal Summation of Divergent Series

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Joint Work with Dr. Robert Dawson

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- If $\mathfrak{S}(a_0 + a_1 + \ldots) = A$, then $\mathfrak{S}(a_1 + a_2 + \ldots) = A - a_0$, and conversely.

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Hardy's Axioms (redux)

A *summation* from R to E (on D) is an R-module homomorphism $\mathfrak{S} : \mathrm{D} \to \mathrm{E}$, such that $\mathfrak{S}(B) = B(1)$ for every $B \in \mathrm{R}[\sigma]$, and $\mathfrak{S}(X) = \mathfrak{S}(\sigma X)$ for each $X \in \mathrm{D}$.

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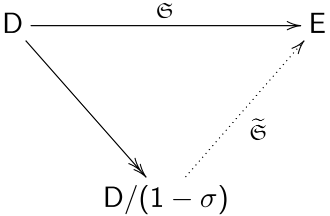

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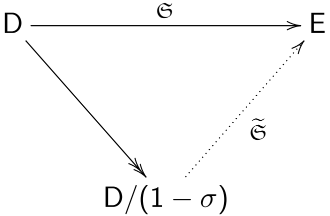

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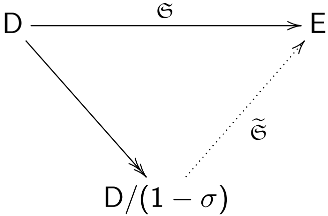

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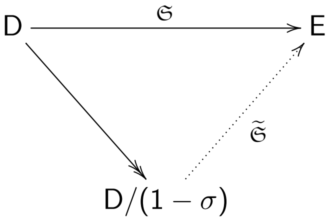

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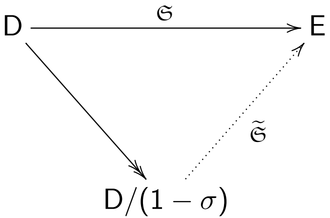

Note D cannot be $\mathrm{R}[[\sigma]]$

Indeed, if $\frac{1}{1-\sigma} = 1 + 1 + 1 + \ldots \in \mathrm{D}$ then $0 = 1 \in \mathrm{E}$, an absurdity

Examples

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Is there a "best" extension of $\mathfrak{S}$?

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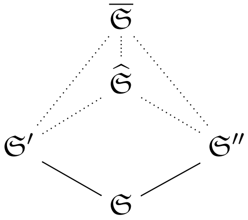

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Theorem (Dawson, 1997)

The summation $(\mathcal{T}\mathrm{D}, \mathcal{T}\mathfrak{S})$ is the fulfillment of $(\mathrm{D}, \mathfrak{S})$.

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Thus $\mathcal{T}\mathfrak{S}_c \neq \mathfrak{S}_c$

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Clearly $\mathbf{MS}(\mathrm{R}, \mathrm{E}) \subseteq \mathbf{wMS}(\mathrm{R}, \mathrm{E}) \subseteq \mathbf{S}(\mathrm{R}, \mathrm{E})$

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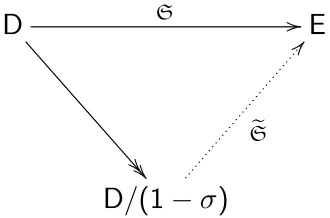

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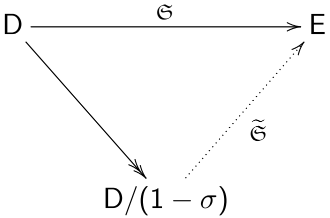

Proposition (2020, Dawson-M.)

Every weakly multiplicative summation $\mathfrak{S}$ has a unique minimal multiplicative extension.

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- $\mathfrak{S}_{\mathbb{A}}$ defined by $\mathfrak{S}_{\mathbb{A}}\left(\sum_n a_n\sigma^n\right) := \lim_{x\nearrow 1}\sum_n a_n x^n$ is multiplicative

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Define $\mathfrak{S} : \mathbb{C}[\sigma] \oplus \mathbb{C}[\sigma] W \oplus \mathbb{C}[\sigma] W^{-1} \to \mathbb{C}$ by

$$\mathfrak{S}\left(B_0 + B_1 W + B_2 W^{-1}\right) := \mathfrak{A}(B_0)$$

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Let $W := \exp(\sigma) = 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\frac{1}{720}+\dots$

So $W^{-1} := \exp(-\sigma) = 1-1+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}+\frac{1}{720}+\dots$

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How do $\mathbf{S}(\mathbb{C},\mathbb{C})$ and $\mathbf{wMS}(\mathbb{C},\mathbb{C})$ compare?

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$$1 = \mathfrak{S}'(1) = \mathfrak{S}'\left(W \cdot W^{-1}\right) = \mathfrak{S}'(W)\,\mathfrak{S}'\left(W^{-1}\right) = 0 \cdot 0 = 0,$$

an absurdity

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Fix a multiplicative summation $(\mathsf{D}, \mathfrak{S}) \in \mathbf{MS}(\mathsf{R}, \mathsf{E})$

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We write $\mathfrak{S}' \supseteq \mathfrak{S}$ if $\mathfrak{S}'$ multiplicatively extends $\mathfrak{S}$. This is an inductive ordering.

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We say $\overline{\mathfrak{S}}$ is the *multiplicative fulfillment* of $\mathfrak{S}$ if $\overline{\mathfrak{S}}$ extends every multiplicative canonical extension of $\mathfrak{S}$.

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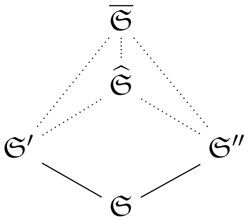

So what's the multiplicative fulfillment of $\mathfrak{S}$?

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We define the *scalar polynomial* $s_X(t)$ *for* X to be 0 if $\mathfrak{S}(P)(t) = 0$, and to be the unique monic scalar multiple of $\mathfrak{S}(P)(t)$ otherwise

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Set $X := \frac{1+\sqrt{1-4\sigma+4\sigma^3}}{2-2\sigma}$
$= 1 - 1 - 2 - 5 - 13 - 36 - 104 - 311 - 955 + \ldots$

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Proposition (Dawson-M., 2020)

Let X be a series, and let $x \in \mathrm{E}$. The following are equivalent:

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Let $(\mathrm{D}', \mathfrak{A}')$ be any extension of $(\mathbb{C}[t], \mathfrak{A})$

If $X_+,\ X_- \in \mathrm{D}'$, then $X_+ + X_- = \frac{1}{1-\sigma} \in \mathrm{D}'$, an absurdity

Absolutely $\mathfrak{S}$-Algebraic Series

Definition

A series X is *absolutely* $\mathfrak{S}$*-algebraic* if X is $\mathfrak{S}'$-algebraic for every multiplicative summation $\mathfrak{S}'$ extending $\mathfrak{S}$

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Let $P(t)$ be a $\mathfrak{S}$-minimal polynomial for X

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As $\deg P(t) = \deg s_Y(t) = 2 < \infty$, we see Y is absolutely $\mathfrak{A}$-algebraic

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Definition

A series X is $\mathfrak{S}$-*univalent (with root* ρ_X*)* if X is $\mathfrak{S}$-algebraic and $s_X(t) = (t - \rho_X)^m$ for some $m \in \mathbb{N}$

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If a series X is $\mathfrak{S}$-univalent and absolutely $\mathfrak{S}$-algebraic, we say X is *absolutely $\mathfrak{S}$-univalent*

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Theorem (Dawson-M., 2020)

The summation $(\mathcal{U}\mathrm{D}, \mathcal{U}\mathfrak{S})$ is the multiplicative fulfillment of $(\mathrm{D}, \mathfrak{S})$.

Extending Weakly Multiplicative Summations

Fix a multiplicative summation $(\mathrm{D}, \mathfrak{S}) \in \mathbf{MS}(\mathrm{R}, \mathrm{E})$

Proof Sketch

Clearly $\mathcal{U}\mathfrak{S}$ is a multiplicatively canonical extension of $\mathfrak{S}$

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- Suppose X is $\mathfrak{S}$-univalent but not absolutely $\mathfrak{S}$-univalent. Then there exists an extension $\mathfrak{S}'$ of $\mathfrak{S}$ for which $s_X(t) = 1$, and we reduce to the previous case

What does $\mathcal{U}\mathfrak{S}_c$ look like?

Example

Set $Z := \frac{3-\sigma+\sqrt{1-6\sigma+5\sigma^2}}{2} = 2 - 2 - 3 - 10 - 36 - 137 - 543 + \ldots$

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Thus $\mathcal{U}\mathfrak{S}_c \neq \mathcal{T}\mathfrak{S}_c$

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What if E is not algebraically closed? What if E is not a field?

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Is there a fruitful algebraic-geometric perspective on all of this?

Thank you for your attention!