Universal norms of p-adic Galois representations and the Fargues-Fontaine curve

On a question by J. Coates & R. Greenberg

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27th September 2020

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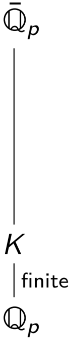

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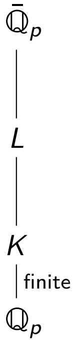

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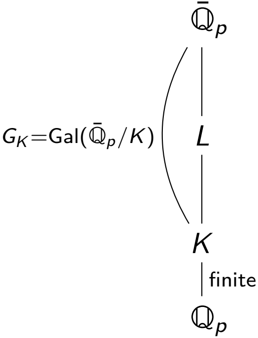

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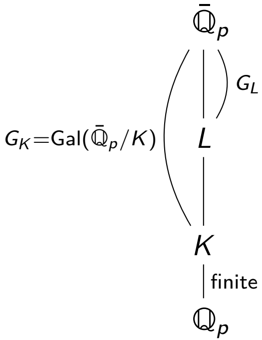

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taking G_L-invariants induces the **Kummer map**:

$$\kappa_L : A(L) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{H}^1(L, A[p^\infty]).$$

Iwasawa theory

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- When L/K finite: BSD-conjecture (Bloch-Kato conjecture (later)).
- When L/K infinite: Iwasawa theory.
  In the case where the completion $\hat{L}$ of L is a perfectoid field: answer by Coates-Greenberg (1996).

Perfectoid fields

Definition (Scholze, 2012)

A complete non-Archimedean field F of residue characteristic p is a **perfectoid field** if its valuation group is non-discrete and the p-th power Frobenius map on $\mathcal{O}_F/(p)$ is surjective.

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4. $K(p^{1/p^\infty})^\wedge$ (non-Galois)

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- Coates-Howson (2001) computation of Euler-Poincaré characteristic of Selmer groups of ordinary elliptic curves.

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If $T = T_p(A) = \varprojlim_{p\times} A[p^n]$, so that $V/T = A[p^\infty]$, then

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Generalisation of Coates & Greenberg's Theorem?

Question (Coates & Greenberg, 1996)

Description of $\mathrm{H}^1_e(L, V/T) \subset \mathrm{H}^1(L, V/T)$ analogous to Coates & Greenberg's Theorem?

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- When $L = K(\mu_{p^\infty})$ and V is de Rham, **Yes** by Berger (2005) and Perrin-Riou (1992,2000,2001).

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Proof relies on:

- Fontaine's theory of almost $\mathbb{C}_p$-representations (2003),
- and the classification of vector bundles over the Fargues-Fontaine curve (2018) (a fundamental result of p-adic Hodge theory).

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5. 4. + [p-cohomological dim. of perfectoid fields $\leqslant 1$]
$\Rightarrow \mathrm{H}^1(L, \hat{\mathcal{A}}) = 0$.

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