Geometric Quadratic Chabauty over Number Fields

joint with Pavel Čoupek, Luciena X. Xiao and Zijian Yao

David Lilienfeldt

McGill University

Québec-Maine Number Theory Conference
September 26, 2020

Preprint available at `https://www.math.mcgill.ca/lilien/`

Motivation

Problem 17 Book VI of Diophantus' *Arithmetica*

Find three squares which when added give a square, and such that the first one is the square-root of the second, and the second is the square-root of the third: $y^2 = x^8 + x^4 + x^2$ $\rightsquigarrow$ Wetherell (1997)

Serre's uniformity question

Does there exist a constant N such that, for any prime $\ell \geq N$ and non-CM elliptic curve E over $\mathbb{Q}$, the residual Galois representation $\bar{\rho}_{E,\ell}$ of E at ℓ is surjective? $\rightsquigarrow$ BDMTV (2019), still open

Both questions can be formulated as asking for solutions to Diophantine equations.

Diophantine Geometry

Let C_K denote a smooth projective curve of genus g defined over a number field K.

Let us assume $C_K(K) \neq \emptyset$.

Diophantine geometry: the study of $C_K(K)$.

- $g = 0$: Hilbert-Hurwitz (1890) $\rightsquigarrow |C_K(K)| = \infty$.
- $g = 1$: Mordell-Weil (1929) $\rightsquigarrow C_K(K) = C_K(K)_{\text{tors}} \oplus \mathbb{Z}^r$.
- $g \geq 2$: Mordell's conjecture (1922), Faltings' theorem (1983), Vojta (1991), Lawrence-Venkatesh (2018) $\rightsquigarrow |C_K(K)| < \infty$.

Question

When $g \geq 2$, how does one determine the finite set $C_K(K)$?

Problem: The proofs of Mordell's conjecture are not effective.

Chabauty-Coleman

Chabauty (1941): $|C_K(K)| < \infty$ if $r := \mathrm{rank}_{\mathbb{Z}} J_K(K) < g$.

$$
\begin{array}{ccccccc}
C_K(K) & \hookrightarrow & & & C_K(K_{\mathfrak{p}}) & \xrightarrow{p\text{-adic integral}} & \\
\downarrow & & & & \downarrow & & \\
J_K(K) & \hookrightarrow & \overline{J_K(K)} & \hookrightarrow & J_K(K_{\mathfrak{p}}) & \xrightarrow{\log} & H^0(J_{K_{\mathfrak{p}}}, \Omega^1)^{\vee}
\end{array}
$$

Idea: bound $\overline{J_K(K)} \cap C_K(K_{\mathfrak{p}})$ inside $\mathfrak{p}$-adic Lie group $J_K(K_{\mathfrak{p}})$.

Coleman (1985): $|C_K(K)| \leq N(\mathfrak{p}) + 2g(\sqrt{N(\mathfrak{p})} + 1) - 1$
when $r < g$, $p > 2g$ and $\mathfrak{p}|p$ is good, unramified.

Improvements by Stoll (2006), Katz–Zureick-Brown (2013).

Chabauty-Kim

Non-abelian Chabauty

Kim (00's): relax the condition "$r < g$" by taking non-abelian unipotent quotients of the fundamental group.

Quadratic Chabauty

Balakrishnan *et al.* (2018): method made effective over $\mathbb{Q}$ for curves satisfying $r < g + \rho - 1$, where $\rho = \operatorname{rank}_{\mathbb{Z}} \mathrm{NS}_{J_{\mathbb{Q}}/\mathbb{Q}}(\mathbb{Q})$.

Geometric quadratic Chabauty

Edixhoven-Lido (2019): new effective approach to Kim's quadratic Chabauty method over $\mathbb{Q}$.

Back to motivation

Problem 17 Book VI of Diophantus' *Arithmetica*

Wetherell (1997): the only positive rational solution to $y^2 = x^8 + x^4 + x^2$ is the one given by Diophantus: $(1/2, 9/16)$.
Method: Chabauty-Coleman (by covering) on the curve $y^2 = x^6 + x^2 + 1$ with $(r = g = 2)$.
Bianchi (2019): same result using quadratic Chabauty.

Serre's uniformity question $\rightsquigarrow N = 37$ (common belief)

Serre (1972), Mazur (1978), Bilu-Parent-Rebolledo (2013).
BDMTV (2019): only rational points on $X_s(13)$ are CM or cusps.
Method: quadratic Chabauty for $X_s(13)$ $(r = g = 3)$.
Non-split case still open.

Generalisation to arbitrary degree d number fields

Siksek ('13): effective Chabauty-Coleman $\rightsquigarrow r \leq d(g-1)$

BBBM ('19): eff. quadratic Chabauty $\rightsquigarrow r \leq d(g-1) + r_2 + 1$

Dogra ('19): quad. Chabauty $\rightsquigarrow r \leq d(g-1) + (\rho - 1)(r_2 + 1)$

CLXY ('20): geom. quad. Ch. $\rightsquigarrow r \leq d(g-1) + (\rho - 1)(r_2 + 1)$

Theorem (Čoupek, L., Xiao, Yao (2020))

Suppose that $r \leq d(g-1) + (\rho-1)(r_2+1)$. Let $\delta := r_1 + r_2 - 1$ and $A := \mathbb{Z}_p\langle z_1, \ldots, z_{\delta(\rho-1)+r}\rangle$. There exists an ideal I of A, such that if $\overline{A} := A/I \otimes \mathbb{F}_p$ is a finite dimensional $\mathbb{F}_p$-vector space, then $|C_K(K)| \leq \dim_{\mathbb{F}_p} \overline{A}$.

Note: the statement is simplified for the sake of exposition.

Thank you for your attention !

Appendix: Simplified strategy

Idea: replace the Jacobian in Chabauty by something bigger.
What: a certain $\mathbb{G}_m^{\rho-1}$-torsor T_K over J_K.

Construction

- T_K arises as a certain pull-back of the $(\rho-1)$-fold self-product of the Poincaré torsor over $J_K \times J_K^\vee$.
- It comes equipped with a lift $C_K \hookrightarrow T_K$ of the Abel-Jacobi map.

Let $K_p := K \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $Y_K := \overline{T_K(K)}^p$.

$$\begin{array}{ccccc} C_K(K) & & \hookrightarrow & & C_K(K_p) \\ \downarrow & & & & \downarrow \\ T_K(K) & \hookrightarrow & Y_K & \hookrightarrow & T_K(K_p). \end{array}$$

Goal: bound and compute $C_K(K_p) \cap Y_K$.