Galois Module Structure of Square Power Classes in Biquadratic Extensions

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Motivation and Background

Motivation

Inverse Galois Problem

If G is a group and K is a field, can we find/parameterize all G-extensions of K?

Kummer theory: if $\text{char}(K) \neq p$ and $\xi_p \in K$:

$$\left\{ \begin{array}{c} \text{Elementary } p\text{-abelian} \\ \text{extensions of K} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \mathbb{F}_p - \text{subspaces} \\ \text{of } \boxed{K^\times / K^{\times p}} \end{array} \right\}$$

Artin-Schreier theory: if $\text{char}(K) = p$:

$$\left\{ \begin{array}{c} \text{Elementary } p\text{-abelian} \\ \text{extensions of K} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \mathbb{F}_p - \text{subspaces} \\ \text{of } \boxed{K / \wp(K)} \end{array} \right\}$$

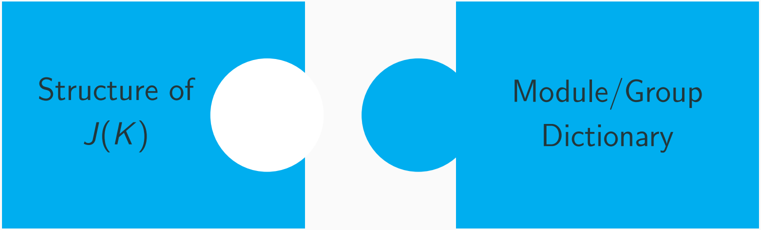
$J(K)$

More structure $\Longrightarrow$ more structure

Proposition (Waterhouse,S-)

If M is an $\mathbb{F}_p$-subspace of $J(K)$, and L/K its extension, then L/F Galois iff M is an $\mathbb{F}_p[\mathrm{Gal}(K/F)]$-module.

In fact, $\mathrm{Gal}(L/F)$ can be computed in terms of module structure of M and some field-theoretic data.



What's been done

$\mathrm{Gal}(K/F)$	Module	Caveats
$\mathbb{Z}/p^n\mathbb{Z}$	$J(K)$	$\emptyset$
$\mathbb{Z}/p^n\mathbb{Z}$	$E^\times/E^{\times p^s}$	$\mathrm{char}(E) \neq p$
$\mathbb{Z}/p\mathbb{Z}$	$H^i(K, \mathbb{F}_p)$	$\xi_p \in K$
$\mathbb{Z}/p^n\mathbb{Z}$	$H^i(K, \mathbb{F}_p)$	$\xi_p \in K$ and embedibility
$\mathbb{Z}/p\mathbb{Z}$	$K_i(K)/p^s K_i(K)$	$\mathrm{char}(K) = p$

The general trend

Modules have far fewer classes of indecomposable modules than one would expect

Punchline: Maximal pro-p quotient of absolute Galois group isn't a generic pro-p group

Corollary

Let $p > 2$. Define $\nu(G, F)$ as number of G-extensions of F. Then $\nu(M_{p^3}, F)$ is

$$(p^2 - 1)\nu(H_{p^3}, F) + \underbrace{\left(\binom{\dim_{\mathbb{F}_p} J(F)}{1}_p - \binom{\dim_{\mathbb{F}_p} \mathfrak{N}}{1}_p\right)}_{\text{“non-embeddable” } \mathbb{Z}/p\mathbb{Z}\text{-extensions of } F} \frac{|J(F)|}{p^2}.$$

Moving away from cyclic extensions

How can we dip our toe into the non-cyclic cases?

Let G be as simple as possible $\rightsquigarrow$ $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

Structure of $K^\times/K^{\times 2}$

Notation

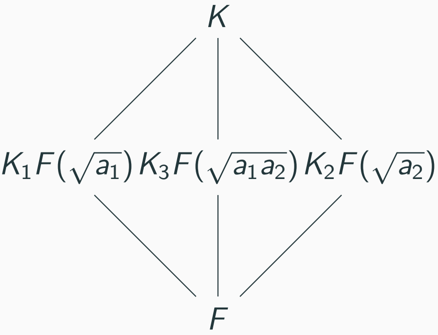
$K = F(\sqrt{a_1}, \sqrt{a_2})$

$\sigma_i(\sqrt{a_j}) = (-1)^{\delta_{ij}}\sqrt{a_j}$

$G = \mathrm{Gal}(K/F) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

$[\gamma] \in K^\times/K^{\times 2}$ is class of
$\gamma \in K^\times$

$[\gamma]_i \in K_i^\times/K_i^{\times 2}$ is class of $\gamma \in K_i$



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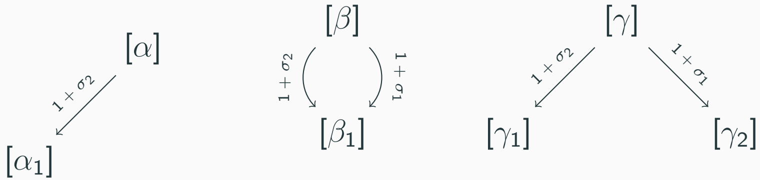


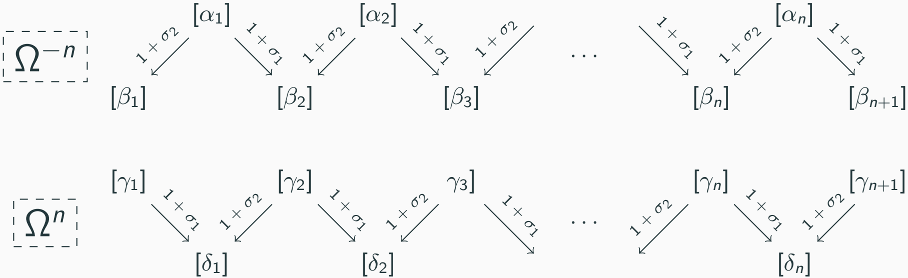
$$[\alpha_1] = [\alpha]^{1+\sigma_2}$$

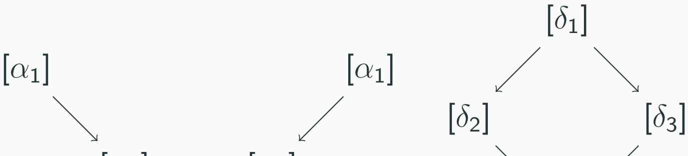
$$\begin{aligned}[\beta_1] &= [\beta]^{1+\sigma_2} \\ &= [\beta]^{1+\sigma_2}\end{aligned}$$

$$\begin{aligned}[\gamma_1] &= [\gamma]^{1+\sigma_2} \\ [\gamma_2] &= [\gamma]^{1+\sigma_1}\end{aligned}$$

A sample of $\mathbb{F}_2[\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}]$-indecomposables

For $n > 0$, there are 2 indecomposables of dimension $2n + 1$





Our module decomposition

Theorem [Chemotti, Mináč, S-, Swallow]

Suppose $\mathrm{char}(K) \neq 2$ and $\mathrm{Gal}(K/F) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Then

$$K^\times/K^{\times 2} \simeq O_1 \oplus Q_0 \oplus Q_1 \oplus Q_2 \oplus Q_3 \oplus Q_4 \oplus X,$$

where

- O_1 is a direct sum of modules isomorphic to Ω^1; and
- for each $i \in \{0, 1, 2, 3, 4\}$, the summand Q_i is a direct sum of modules isomorphic to $\mathbb{F}_2[G/H_i]$; and
- X is isomorphic to one of the following: $\{0\}, \mathbb{F}_2, \mathbb{F}_2 \oplus \mathbb{F}_2, \Omega^{-1}, \Omega^{-2}$, or $\Omega^{-1} \oplus \Omega^{-1}$.

Sketch of proof

Playbill

Lemma (Exclusion lemma)

If $U, V \subseteq W$ are $\mathbb{F}_2[G]$-modules, then

$$U \cap V = \{0\} \iff U^G \cap V^G = \{0\}$$

Strategy: Focus on $(K^\times/K^{\times 2})^G = [F^\times]+??$

Act I: Build a big module Y with $Y^G = [F^\times] \subseteq (K^\times/K^{\times 2})^G$

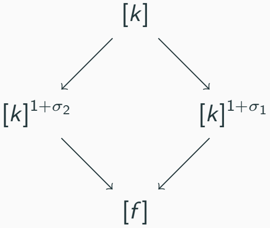
Act II: Build a big module X "over" a complement to $[F^\times]$

Act III: Show $X + Y$ spans

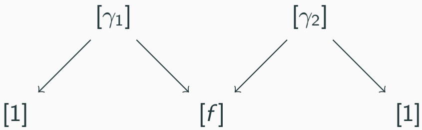
Sketch of proof

Act I: Building over $[F^\times]$

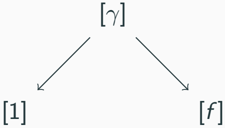
Act I: maximize preimages, minimize generators



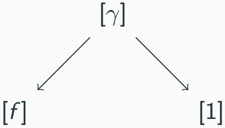
$\mathfrak{A} = \{[f] : \exists [k] \ni \dots\}$



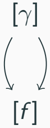
$\mathfrak{B} = \{[f] : \exists [\gamma_1], [\gamma_2] \ni \dots\}$



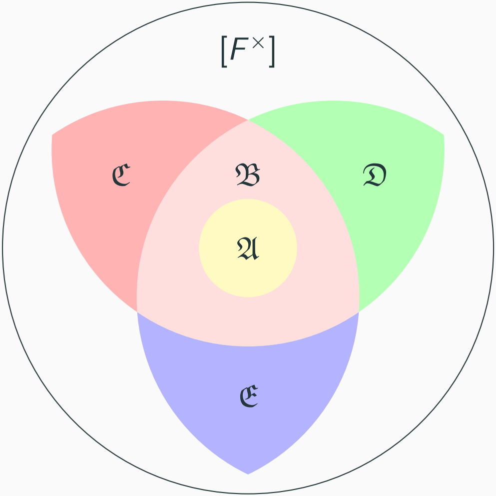
$\mathfrak{C} = \{[f] : \exists [\gamma] \ni \dots\}$



$\mathfrak{D} = \{[f] : \exists [\gamma] \ni \dots\}$



$\mathfrak{E} = \{[f] : \exists [\gamma] \ni \dots\}$


$[F^\times]$
$\mathfrak{C}$
$\mathfrak{B}$
$\mathfrak{D}$
$\mathfrak{A}$
$\mathfrak{E}$

Act I: Conclusion

Proposition

There exists a submodule Y whose fixed part is $[F^\times]$, and which is a direct sum of modules isomorphic to

- $\mathbb{F}_2[G/H_i]$ for $i \in \{0, 1, 2, 3, 4\}$
- Ω^1

Sketch of proof

Act II: Filling out $(K^\times/K^{\times 2})^G$

Act II: WTF

Lemma (Whether 'tis $[f]$)

For $[\gamma] \in (K^\times/K^{\times 2})^G$, the following are equivalent:

- $[\gamma] \in [F^\times]$
- $\mathrm{Gal}(K(\sqrt{\gamma})/F) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
- $[\gamma] \in \bigcap_{i=1}^{3} \ker \left(K^\times/K^{\times 2} \xrightarrow{N_{K/K_i}} K_i^\times/K_i^{\times 2} \right)$

Act II: How we build a complement

$[F^\times]$ is kernel of $T : J^G \to \bigoplus_{i=1}^3 (K_i^\times \cap K^{\times 2})/K_i^{\times 2}$ given by

$$T([\gamma]) = \big([N_{K/K_1}(\gamma)]_1, [N_{K/K_2}(\gamma)]_2, [N_{K/K_3}(\gamma)]_3\big)$$

Goal: Find "big" preimage for $\mathrm{im}(T)$ that has trivial intersection with $[F^\times]$

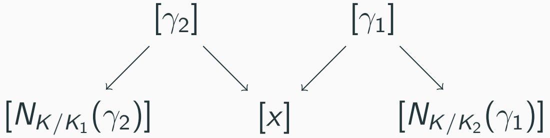
What we get depends on $\mathrm{im}(T)$

Act II: An example

Suppose $[x] \in \left([N_{K/K_1}(K^\times)] \cap [N_{K/K_2}(K^\times)] \cap J^G\right) \setminus \ker(T)$

$\rightsquigarrow$ exists $[\gamma_1], [\gamma_2]$ so that $[N_{K/K_i}(\gamma_i)] = [x]$

$\rightsquigarrow$ $\dim\left(T(\{[x], [N_{K/K_1}(\gamma_2)], [N_{K/K_2}(\gamma_1)]\})\right) = 3$



Ω^{-2}

Act II: Another example

Suppose that $\operatorname{im}(T) = \{([1]_1, [v]_2, [w]_3\}$

$\rightsquigarrow$ solvability of certain "small" Galois embedding problems

$\rightsquigarrow$ solvability of particular "large" Galois embedding problem

$\rightsquigarrow$ exists $[\gamma]$ so that $\mathrm{im}(T) = T\left(\{[N_{K/K_1}(\gamma)], [N_{K/K_2}(\gamma)]\}\right)$

$$[\gamma] \swarrow [N_{K/K_1}(\gamma)] \qquad [\gamma] \searrow [N_{K/K_2}(\gamma)]$$

Ω^{-1}

Act II: Constructing X

Proposition

Suppose $\{\mathrm{im}(T)\} \neq \{[1]_1, [1]_2, [1]_3\}$. Then there exists $X \in J(K)$ with $T(X^G) = \mathrm{im}(T)$, so that X is isomorphic to

$$\begin{cases} \mathbb{F}_2, & \text{if } \dim_{\mathbb{F}_2}(\mathrm{im}(T)) = 1 \\ \Omega^{-1}, & \text{if } \mathrm{im}(T) \text{ is a "coordinate plane"} \\ \mathbb{F}_2 \oplus \mathbb{F}_2, & \text{if } \mathrm{im}(T) \text{ is a "non-coordinate plane"} \\ \Omega^{-2}, & \text{if } T([N_{K/K_1}(K^\times)] \cap [N_{K/K_2}(K^\times)] \cap J^G) \text{ nontrivial} \\ \Omega^{-1} \oplus \Omega^{-1}, & \text{else.} \end{cases}$$

Note: in final case $\dim(X \cap [F^\times]) = 1$. Requires small Y tweak.

Sketch of proof

Act III: Putting it all together

Act III: Gotta catch 'em all

$X + Y = X \oplus Y$ by "exclusion lemma". Do they span?

Case 1: Suppose $\langle[\gamma]\rangle \simeq \mathbb{F}_2$

$\rightsquigarrow$ Can assume $T([\gamma]) = ([1]_1, [1]_2, [1]_3)$ by X

$\rightsquigarrow$ We picked up all of $[F^\times]$ in Y^G ✓

Act III: Still gotta catch 'em all

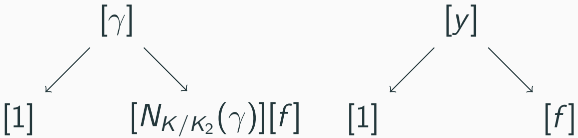
Case 2: Suppose $\langle[\gamma]\rangle \simeq \mathbb{F}_2[G/H_1]$.

$\rightsquigarrow$ Can prove $T([N_{K/K_2}(\gamma)]) = ([1]_1, [1]_2, [1]_3)$

$\rightsquigarrow$ $[N_{K/K_2}(\gamma)] = [f] \in \mathfrak{C}$

$\rightsquigarrow$ $\exists [y] \in Y$ with same images under $1 + \sigma_i$

$\rightsquigarrow$ $\langle[\gamma]/[y]\rangle \simeq \{[1]\}$ or $\langle[\gamma]/[y]\rangle \simeq \mathbb{F}_2$ ✓



Act III: Almost caught 'em all

Case 3: Suppose that $\langle[\gamma]\rangle \simeq \Omega^1$

$\rightsquigarrow$ Can assume $\langle[\gamma]\rangle^G \subseteq \ker(T)$ by X's construction

$\rightsquigarrow$ Lemma: $[F^\times] \cap [N_{K/K_1}(K^\times)] \subseteq \mathfrak{D} \cdot \mathfrak{E}$

$\rightsquigarrow$ Can "cut down" to a module type already checked

Act III: Cutting the module

$$
\begin{array}{ccccc}
[\gamma_{1,3}] & & [\gamma_{1,2}] & & [\gamma] \\
\curvearrowleft\ \curvearrowright & \swarrow & & \swarrow & \searrow \\
[f_{1,3}] & [f_{1,2}] & = [N_{K/K_1}(\gamma)] & & [N_{K/K_2}(\gamma)] \\
\in \mathfrak{E} & \in \mathfrak{D} & \in [F^\times] & &
\end{array}
$$

Then $([\gamma][\gamma_{1,3}][\gamma_{1,2}])^{1+\sigma_2} = [1]$

$\rightsquigarrow$ so $\langle [\gamma][\gamma_{1,3}][\gamma_{1,2}] \rangle$ is some previous case.

Thank you!