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Modular Properties of Elliptic Genus of Hilbert Scheme of Points on $\mathbb{C}^2$

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Abstract

The elliptic genus of a compact complex manifold can be defined as the integral over M of some multiplicative class. If the first Chern class of M vanishes, then the elliptic genus of M is a Jacobi form. We will discuss the elliptic genus for non-compact manifold, in particular, the Hilbert scheme of points on $\mathbb{C}^2$ which we will be denoted by $\text{Hilb}^n[\mathbb{C}^2]$ and we will also discuss its recursive structure in terms of Hecke operators.

Modular Forms

A modular form of weight $w \in \mathbb{Z}$ is a holomorphic function on H and at the cusp P_∞ satisfying the following functional equation

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^w f(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

Its Fourier series is of the form $f(\tau) = \sum_{n=0}^{\infty} a(n)e^{2\pi i n\tau}$. If $a(0) = 0$, then the modular form f is referred to as the weight w cusp form.

Jacobi Forms: Let $w \in \mathbb{Z}$, $\ell \in \mathbb{Z}^+, \tau \in H, z \in \mathbb{C}$. A Jacobi form is a holomorphic function $J : H \times \mathbb{C} \to \mathbb{C}$ which satisfies the following modular and elliptic properties:

$$\begin{aligned} J\left(\tfrac{a\tau+b}{c\tau+d}, \tfrac{z}{c\tau+d}\right) &= (c\tau+d)^w e^{\frac{2\pi i \ell c z^2}{c\tau+d}} J(\tau, z) \\ J(\tau, z+u\tau+v) &= e^{-2\pi i\ell(u^2\tau+2uz)} J(\tau, z), \end{aligned}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $u, v \in \mathbb{Z}$. It has Fourier expansion of the form

$$J(\tau, z) = \sum_{n,\ell\in\mathbb{Z}} a(n, s)e^{2\pi i(n\tau+sz)}$$

$\text{Hilb}^k[\mathbb{C}^2]$

As a set Hilbert scheme of points on $\mathbb{C}^2$ is the set of ideals in the polynomial ring $\mathbb{C}[x_1, x_2]$ such that the as a quotient space $\mathbb{C}[x_1, x_2]/I$ is isomorphic to a k-dimensional vector space over $\mathbb{C}$ i.e.,

$$\text{Hilb}^k[\mathbb{C}^2] = \{I \subset \mathbb{C}[x_1, x_2] \mid \dim(\mathbb{C}[x_1, x_2]/I) = k\}.$$

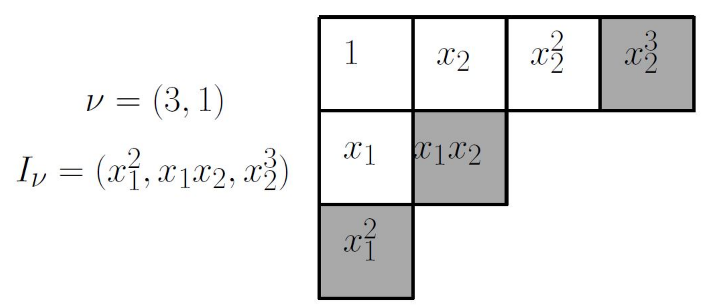
Example: The ideal $I =< x_1^k, x_2 >$ is an ideal of colength k in $\mathbb{C}[x_1, x_2]$
The action of torus $\mathbb{T}^2$ on $\mathbb{C}^2$ induces an action on ideals in $\mathbb{C}[x_1, x_2]$ in the following way,

$$(t_1, t_2) \cdot I = \{(f(t_1^{-1}x_1, t_2^{-1}x_2)) \mid f(x_1, x_2) \in I\}.$$

Since, this action preserves the length of the ideal, therefore, it induces the torus action on $\text{Hilb}^k[\mathbb{C}^2]$. The fixed points of the torus action on $\text{Hilb}^k[\mathbb{C}^2]$ are the monomial ideals. According to Ellingsrud and Strømme there is a one to one correspondence between the fixed points monomial ideals of length k in $\mathbb{C}[x_1, x_2]$ and the partitions ν of k.

$$\nu \to I_\nu = \{x_1^r x_2^s \mid (r, s) \notin \nu\}$$
$$I \to \nu(I) = \{(r, s) \mid x_1^r x_2^s \notin I\}$$

The basis set of vector space $\mathbb{C}[x_1, x_2]/I$ is the set $B_\nu = \{x_1^r x_2^s \mid (r, s) \in \nu\}$.



The tangent space at fixed point I of $\mathbb{T}^2$ action and corresponding to the Young diagram Y is $T_I(\text{Hilb}^k[\mathbb{C}^2]) = \sum_{\nu\in Y}(t_1^{\ell(\nu)+1}t_2^{-a(\nu)} + t_1^{-\ell(\nu)}t_2^{a(\nu)+1})$.
Here, $a(\nu)$ is the arm length and $\ell(\nu)$ is the leg length of the partition.

Elliptic Genus

Let M be a compact complex manifold dimension d. Consider the formal power series in $q = e^{2\pi i\tau}, y = e^{2\pi iz}$ whose coefficients are vector bundles

$$E_{q,y} = y^{-d/2} \bigotimes_{n=1}^{\infty} \Big(\wedge_{-y^{-1}q^{n-1}} E^* \otimes \wedge_{-yq^n} E \otimes S_{q^n} T_M^* \otimes S_{q^n} T_M \Big)$$

where E^* is dual to the vector bundle E and T_M^* is dual to the tangent bundle T_M.
$\wedge_y E = \sum_{k\geq 0}(\wedge^k E)y^k$ and $S_y E = \sum_{k\geq 0}(S^k E)y^k$,
where $\wedge^k$ and S^k are the k-th exterior product and symmetric product respectively. The elliptic genus is defined as

$$\chi_{\text{ell}}(M) = y^{-d/2} \int_M ch(E_{q,y})Td(T_M)$$

When $\dim M = \dim E = d$, then in term of chern roots it simplifies to the following integral

$$\chi_{\text{ell}}(M) = y^{-d/2} \int_M \prod_{i=1}^{d} x_i \frac{\theta_1(\tau, z + \frac{x_i}{2\pi i})}{\theta_1(\tau, \frac{x_i}{2\pi i})}$$

In case, M is non-compact manifold and admits a torus action with r fixed points, then we can integrate the integral using equivariant localization

$$\chi_{\text{ell}}(M) = y^{-d/2} \sum_{j=1}^{r} \prod_{i=1}^{d} \frac{\theta_1(\tau, z + \frac{x_{i,j}}{2\pi i})}{\theta_1(\tau, \frac{x_{i,j}}{2\pi i})}$$

Here, $\theta_1(\tau, m)$ is an odd Jacobi form

$$\theta_1(\tau, z) = -iq^{\frac{1}{8}}y^{\frac{1}{2}} \prod_{n=1}^{\infty}(1-q^n)(1-q^n y)(1-q^{n-1}y^{-1})$$

Recursive structure

Consider the generating function using the elliptic genus of $\text{Hilb}^k[\mathbb{C}^2]$

$$Z(\tau, t, z, \epsilon_{1,2}) = \sum_{k\geq 0} Q^k \chi_{ell}(\text{Hilb}^k[\mathbb{C}^2]),$$

where $Q = e^{-t}$. This generating function is basically the Nekrasove Partition function, the free energy associated to this partition function is

$$F(\tau, t, z, \epsilon_{1,2}) = \log\Big(Z(\tau, t, z, \epsilon_{1,2}\Big)$$

The free energy has the summation of the following form

$$F(\tau, t, z, \epsilon_{1,2}) = \sum_{k>0} Q^k G^{(k)}(\tau, m, \epsilon_{1,2})$$

where $G^{(k)}$ is a Jacobi form. The coefficient of Q, $G^{(1)}$ has the following expression

$$G^{(1)}(\tau, z, \epsilon_{1,2}) = \frac{\theta_1(\tau, z + \epsilon_-)\theta_1(\tau, z - \epsilon_-)}{\theta_1(\tau, \epsilon_+ + \epsilon_-)\theta_1(\tau, \epsilon_+ - \epsilon_-)}.$$

Here, $\epsilon_+ = \frac{\epsilon_1+\epsilon_2}{2}$ and $\epsilon_- = \frac{\epsilon_1-\epsilon_2}{2}$. It was shown by [1] that all $G^{(k)}$ can be written in term of Hecke of $G^{(1)}$

$$F(\tau, t, z, \epsilon_{1,2}) = \sum_{k>0} Q^k H_k(G^{(1)}(\tau, z, \epsilon_{1,2}))$$

where H_k is the linear operator which preserves the weight of the Jocbi form and is defined as

$$(H_k J)(\tau, z) = k^{w-1} \sum_{\substack{ad=k \\ a>0}} \sum_{0\leq b<d} \frac{1}{d^w} J\left(\frac{a\tau+b}{d}, az\right)$$

References

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