Explicit coverings of K3 surfaces by the square of a curve

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Curves and surfaces

To introduce this talk, let us ask the question: which varieties can be described in terms of curves?

That is, given V an irreducible algebraic variety over $k = \bar{k}$, are there curves $C_1, C_2, \ldots, C_n$ and a dominant rational map $\prod_{i=1}^{n} C_i \dashrightarrow V$?

Taking $n = \dim V$ and all C_i equal does not change the answer.

A more arithmetical version of the question: for which V can we describe all local zeta functions of V mod p uniformly in terms of those of curves?

This does not sound like a question that will ever be answered, so let's restrict to $\dim V = 2$.

Which surfaces to look at?

This is a birationally invariant property of V, so we may as well assume that V is minimal.

Serre showed that the answer is no in general (surfaces contained in abelian varieties).

If V is a ruled surface, then the answer is obviously yes, since $V \sim C \times \mathbb{P}^1$.

Let's look at surfaces of Kodaira dimension 0.

Kodaira dimension 0

1. If V is an abelian surface, then V is isogenous to $\mathrm{Jac}\, C$ for some curve C of genus 2, so V is dominated by $C \times C$.
2. If V is bielliptic, then certainly V is dominated by a product of curves.
3. If V is an Enriques surface, we should probably start by looking at the K3 surface that covers it.
4. If V is a K3 surface, then things are unclear.

I will talk about case 4 today.

Reminders on K3 surfaces

A *K3 surface* is a smooth surface S with trivial canonical divisor and fundamental group.

Standard examples:

1. A double cover of $\mathbb{P}^2$ branched along a smooth sextic;
2. a smooth quartic in $\mathbb{P}^3$;
3. a smooth intersection of a quadric and a cubic in $\mathbb{P}^4$;
4. a smooth complete intersection of three quadrics in $\mathbb{P}^5$.

There is a family of K3 surfaces of degree $2d - 2$ in $\mathbb{P}^d$ for all $d > 2$.

One can allow certain mild singularities as well.

Covering K3 surfaces by curves

Some K3 surfaces are easily seen to be covered by curves.

For example, let A be an abelian surface. The *Kummer surface* $A/\pm 1$ is a K3 surface, and it is dominated by $C \times C$, where $C \times C$ also dominates A.

However, most K3 surfaces are not Kummer surfaces.

In a paper from 1988, Paranjape gave an interesting construction, building on work of Schoen.

I will not describe it in detail, but he shows that a double cover of $\mathbb{P}^2$ branched along six lines is covered by the square of a curve of genus 5.

This talk

Our main goal in this talk is to describe the geometry of a construction inspired by Paranjape's.

This will give us a new family of K3 surfaces that are covered by the square of a curve.

We analyze the map of moduli spaces in our situation, showing that it is a birational equivalence. The same methods would probably apply in Paranjape's situation.

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Construction

Let E be an elliptic curve with points $p_0, \ldots, p_3$. Let E_3 be an unramified cover of degree 3, and let D_3 be a double cover branched on the p_i.

Let $C_3 = D_3 \times_E E_3$. Then $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ acts on C_3, and $G \wr \mathcal{S}_2$ acts on $C_3 \times C_3$.

Let $W = (C_3 \times C_3)/\langle(\gamma, \gamma^{-1}), \sigma\rangle$, where σ switches the factors.

A quotient of W

The map $C_3 \times C_3 \to E_3$ factors through W, which makes W an elliptic surface with 18 fibres of type I_2.

It turns out that W has an involution λ that acts as negation on the base and fixes one fibre pointwise. Let $K = W/\lambda$: then K is an elliptic surface with nine I_2 and one I_0^* fibre and so Euler characteristic 24.

K is a K3 surface.

The Picard lattice of K

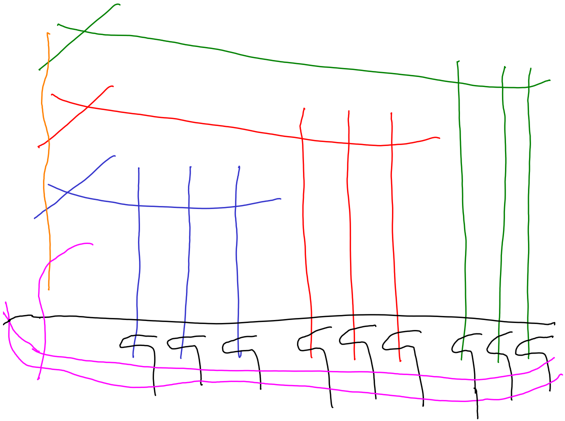
The surface K has many other fibrations. One rather nice one has reducible fibre types I_0^*, I_0^*, I_0^*, I_3 and trivial Mordell-Weil group. The next two slides illustrate the relation between this fibration and the one already described.

The transcendental lattice $H^2(K)/\operatorname{Pic} K$ is isomorphic over $\mathbb{Q}$ to $V \simeq \wedge^2 \mathbb{Q}(\sqrt{-3})^2$. This is connected with the exceptional isogeny

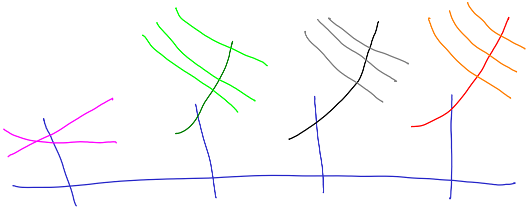
$$SU(U + U, \mathbb{Q}(\sqrt{-3})) \to SO(U \oplus U \oplus \langle -2 \rangle \oplus \langle -6 \rangle).$$

This is also related to work of Garbagnati and Sarti: our K3s are related to K3s in $\mathbb{P}^4$ with 15 ordinary double points.

Going from D4 and nine A1 to A2 and three D4



Going from A2 and three D4 to D4, nine A1, full
2-torsion, and a section of infinite order
(zero section and zero components of A1's not shown)



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The first step back

K comes with an elliptic fibration. Also, the I_1 fibres come in three sets of three.

So, given K, we have a point on $\mathbb{P}^1$ (the location of the I_0^*) and three sets of three points on $\mathbb{P}^1$. This gives us a point on a quotient of $\overline{\mathcal{M}}_{0,10}$, the moduli space of stable curves of genus 0 with 10 marked points.

There must be relations here. In fact the image consists of curves with a map of degree 3 to $\mathbb{P}^1$ such that the first point is in a ramified fibre and the three sets of three are all fibres. Let the images of these points be P_0, Q_1, Q_2, Q_3.

The second step back (1)

This sounds like we just have four points on the image $\mathbb{P}^1$, but in fact we have three more: the images of the other points of ramification. Let these be P_1, P_2, P_3.

We can use these data to recover an elliptic curve and four points on it. Indeed, let E be the double cover of $\mathbb{P}^1$ branched at the P_i, and let $\pm B_i$ be the inverse images of the Q_i. Take the point above P_0 as the origin.

There are four points $R_0, \ldots, R_3$ on E such that $\{R_i - R_j : i \neq j\} = \{\pm B_k \pm B_\ell : k \neq \ell\}$, and these are unique up to translation and negation. The divisor $\sum(R_i) - 2(O) - 2(\sum B_i)$ is principal.

The second step back (2)

We have a triple cover of $\mathbb{P}^1$ whose branch points we have taken as the branch points of a double cover.

This is exactly like what happens when we start with a 3-isogeny of elliptic curves $E' \to E$ and pass to the quotients by ± 1: we obtain a map $E'/\pm 1 \to E/\pm 1$, which is a map of degree 3 ramified at the points under the 2-torsion points of E.

So we can pull back to E and obtain an unramified cover $E_3 \to E$ of degree 3.

(There are 4 triple covers of $\mathbb{P}^1$ with four given points of ramification. The choice turns out to be essentially the same as that involved in choosing the double cover $D_3 \to \mathcal{E}$.)

Three moduli spaces

So we have three moduli spaces: one of curves of genus 1 with additional data, one of K3 surfaces with a chosen fibration, and one of curves of genus 0 with some marked points. There is a map from each of these to the next. They are all irreducible of dimension 4.

With some careful analysis one can show that the composition of the three maps is the identity, and conclude that all of them are birational equivalences. (This question is not considered by Paranjape, but it seems likely that a similar procedure would apply.)

In other words, a typical K3 surface of this type comes from a unique curve of genus 7 of the correct type.

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Counting points

Most concretely, we have shown that the ℓ-adic Galois representations for K are quotients of those for $C_3 \times C_3$. In other words, if we have a description of the local zeta function of C_3 at p we can use it to determine that of K. (Though perhaps still complicated, since there are other quotients that need to be subtracted off.)

Kuga-Satake-Deligne and Hodge

This result gives a concrete illustration of some deep conjectures and constructions. In particular, the Hodge conjecture is unknown in general for abelian fourfolds of Weil type (though in our case it was known by work of Schoen). Our results make it explicit in the special case of an endomorphism ring $\mathbb{Z}[\zeta_3]$. We have also clarified the relation between K and its *Kuga-Satake abelian variety* in this setting.

A compelling question

Laterveer, noting that all K3 double covers of $\mathbb{P}^2$ with 15 nodes are covered by curves, asked whether the same is true for all K3 surfaces with 15 nodes.

The problem can now be solved for degree 6.

We have another construction that should give the result for degree 4. It is more difficult, essentially because $\mathbb{Q}(\sqrt{-2})$ is not a cyclotomic field whereas $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$ are. But the basic idea is similar: we take an unramified cyclic cover of degree 8 of a curve of genus 3 and pass to a suitable quotient of its square.

Are there constructions analogous to these that would give the same result in other cases?

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Thank you

Thank you for your attention.

For more details please see
https://arxiv.org/abs/2009.07807.

Are there any questions?