An Explicit Example of the Hecke Operator

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The 2019 Maine-Quebec Number Theory Conference

October 9, 2019

Modular forms

Definition

A *modular form of weight* k $(k \geq 2)$ for a congruence subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function $f : H = \{\tau \in \mathbb{C}, Im(\tau) > 0\} \to \mathbb{C}$ such that

$$f[\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)]_k(\tau) = f(\frac{a\tau + b}{c\tau + d})(c\tau + d)^{-k} = f(\tau) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$$

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If $a_0 = 0$ in Fourier expansion of $f[\alpha]_k$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. then f is a cusp form. Let $S_k(\Gamma_0(N))$ be the space of cusp form

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$$U_p f(\tau) = \sum_{n \geq 0} \left(a_{np}(f)\right) q^n \text{when p devides N}$$

Eigenforms

- There exist a basis of $S_k(\Gamma_0(N))$ consisting of eigenfunctions for all Hecke operators, called *eigenforms*

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- Normalize the eigenforms so that a_1 =1. We have

$$T_p f = a_p f.$$

- The p-adic valuation of a_p is called the *slope of the eigenforms*.

Gouvea Conjecture

Conjecture (Gouvêa)

Let μ_k denote the density measure on $[0,1]$ given by the set of U_p-slopes on $S_k(\Gamma_0(Np))$, which are normalized by being divided by $k-1$ and are counted with multiplicities. As $k \to \infty$, μ_k converges to the sum of the delta measure of mass $\frac{p-1}{p+1}$ at $\frac{1}{2}$, and the uniform measure on $[0, \frac{1}{p+1}] \cup [\frac{p}{p+1}, 1]$ with total mass $\frac{2}{p+1}$.

Jacquet-Langlands Correspondence

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The Jacquet-Langlands correspondence allows translating the problem about modular form to autormorphic forms on definite quaternion algebras. The upshot is that we can work with these forms "combinatorically"

Definite quaternion algebra

Let D be the quaternion algebra over $\mathbb{Q}$ that ramifies exactly at 2 and ∞,explicitly

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We fix an isomorphism such that

$$1 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, j \to \begin{pmatrix} \nu_5 & 0 \\ 0 & -\nu_5 \end{pmatrix} \text{ and } k \to \begin{pmatrix} 0 & -\nu_5 \\ -\nu_5 & 0 \end{pmatrix}$$

where ν_5 is a square root of -1 in $\mathbb{Q}_5$: $\nu_5 = 2 + 5 + ...$

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We consider the following open compact subgroup of $D_f^\times$:

$$U = D^\times(\mathbb{Z}_2) \times \prod_{\ell \neq 2,5} \mathrm{GL}_2(\mathbb{Z}_\ell) \times \begin{pmatrix} \mathbb{Z}_5^\times & \mathbb{Z}_5 \\ 5\mathbb{Z}_5 & 1 + 5\mathbb{Z}_5^\times \end{pmatrix}$$

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Remark: there is a natural isormorphism

$$\begin{aligned} D^\times \times U &\xrightarrow{\cong} D_f^\times \\ (\delta, u) &\longmapsto \delta u \end{aligned}$$

Overconvergent automorphic forms

Consider the weight k overconvergent automorphic forms:
$S_k^{D,\dagger}(U) := \{\varphi : D^\times \backslash D_f^\times \to \mathbb{Q}_5\langle z\rangle;\ \text{s.t. } \varphi(gu) = \varphi(g)|_{u_5} \text{ for all } u \in U\}$

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where the action $\varphi(g)\|_{u_5}$ defined as for $f(z)$on $\mathbb{Q}_5\langle z\rangle$:

$$f(z)\Big|_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = (cz+d)^{k-2} f\big(\frac{az+b}{cz+d}\big)$$

From the isomorphism $D^\times \times U \cong D_f^\times$, we can identify the space explicitly as follows

$$S_k^{D,\dagger}(U) \longrightarrow \mathbb{Q}_5\langle z\rangle$$
$$\varphi \longmapsto \varphi(1)$$

U_5-action

We choose a decomposition of the double coset:

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for example with $v_j = \left(\begin{smallmatrix} 5 & 0 \\ 5j & 1 \end{smallmatrix}\right)$,(at the component at 5) and define

$$U_p(\varphi) := \sum_{j=0}^{4} \varphi|_{v_j}, \quad \text{with } (\varphi|_{v_j})(g) := \varphi(g v_j^{-1})\|_{v_j}.$$

This definition does not depend on the choice of the coset representatives v_j

U_5 action on $\mathbb{Q}_5\langle z\rangle$

Expressing the U_p operator in terms of the isomorphism $S_k^{D,\dagger}(U) \cong \mathbb{Q}_5\langle z\rangle$,it takes the forms of

$$f(z) \mapsto \sum_{i=1}^{5} f(z)|_{\delta_i}$$

where the matrices δ_i turn out to be global elements in $D(\mathbb{Q})$ (but this fact will not help our local argument later), they are

$$\begin{pmatrix} 2+\nu_5 & 0 \\ 0 & -2-\nu_5 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1-3\nu_5 & 1+3\nu_5 \\ -1+3\nu_5 & 1+3\nu_5 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1-3\nu_5 & 3-\nu_5 \\ -3-\nu_5 & 1+3\nu_5 \end{pmatrix},$$
$$\frac{1}{2}\begin{pmatrix} 1-3\nu_5 & -3+\nu_5 \\ 3+\nu_5 & 1+3\nu_5 \end{pmatrix}, \quad \text{and} \quad \frac{1}{2}\begin{pmatrix} 1-3\nu_5 & -1-3\nu_5 \\ 1-3\nu_5 & 1+3\nu_5 \end{pmatrix}.$$

We need to make explicit what the action of the 5 matrices on the Banach space $\mathbb{Q}_5\langle z\rangle$

Matrix

We use the idea of Jacob in his thesis. Let $(P_{i,j})_{i,j=0,1,\dots}$ denote the matrix for the operator U_p on $\mathbb{Q}\langle z\rangle$ with respect to the power basis $1, z, z^2, \dots,$ then we have an explicit expression of the generating series:

$$H_P(x,y) := \sum_{i,j\geq 0} P_{i,j} x^i y^i = \sum_{i=1}^{5} \frac{(c_i x + d_i)^{k-1}}{c_i x + d_i - a_i x y - b_i y},$$

where $\delta_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$.

Explicit entries of the matrix

$$P_{ij} = \begin{cases} (\frac{1+3\nu_5}{2})^{k-i-2}(\frac{-1+3\nu_5}{2})^{i}\left(4\sum_{n=0}^{i}(-1)^{i-n}\binom{j}{n}\binom{k-j-2}{i-n} \right. & \\ \left. +(-1)^{k-1}(1-\nu_5)^{k-j-2}\nu_5^{j}\right) & \text{if } i=j \\ (\frac{1+3\nu_5}{2})^{k-i-2}(\frac{-1+3\nu_5}{2})^{i}\left(4\sum_{n=0}^{min\{i,j\}}(-1)^{i-n}\binom{j}{n}\binom{k-j-1}{i-n}\right) & \text{if } i \neq j \text{ but } 4|i-j \\ 0 & \text{otherwise} \end{cases}$$

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Remark: $P_{i,j} = 0$ unless $4|i-j$

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From now on, I will focus on $U_p^{(0)} = \left(P_{4i,4j}\right)_{i,j=0,1,\ldots}$.

Corank of the first nxn principle minor

The following table give the conrak of the first $n \times n$-principle minor of the U_p matrix, as the weight takes different values $k = 4k_0 + 2$

k_0	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
2x2	1	1	1		1										
3x3		1	2	1	2	1	1	1	1		1				
4x4			1	1	3	2	2	2	2	1	2	1	1	1	1
5x5					2	2	3	3	3	2	3	2	2	2	2
6x6					1	1	2	3	4	3	4	3	3	3	3
7x7							1	2	3	3	5	4	4	4	4
8x8								1	2	2	4	4	5	5	5
9x9									1	1	3	3	4	5	6
10x10											2	2	3	4	5
11x11											1	1	2	3	4

Main Theorem

Theorem

Let $d^{\mathrm{unr}}_{4k_0+2}$ *and* $d^{\mathrm{Iw}}_{4k_0+2}$ *be dimensions of* $S^D_{4k_0+2}(\mathrm{GL}_2(\mathbb{Z}_5))$ *and* $S^D_{4k_0+2}(U)$*, respectively, then if* $d^{\mathrm{unr}}_{4k_0+2} < n < d^{\mathrm{Iw}}_{4k_0+2} - d^{\mathrm{unr}}_{4k_0+2}$*, then the rank of nxn principle minor is at most*

$$\max\{d^{\mathrm{unr}}_{4k_0+2}, 2n + d^{\mathrm{unr}}_{4k_0+2} - d^{\mathrm{Iw}}_{4k_0+2}\}.$$

Sketch of the proof

Consider the map between $S^D_{4k_0+2}(\mathrm{GL}_2(\mathbb{Z}_5))$ and $S^D_{4k_0+2}(U)$,

$$i : S^{D,\dagger}_{4k_0+2}(\mathrm{GL}_2(\mathbb{Z}_5)) \longrightarrow S^{D,\dagger}_{4k_0+2}(U)$$

$$\varphi(x) \longmapsto \varphi(x\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1})|_{\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)}$$

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$$proj : S^{D,\dagger}_{4k_0+2}(U) \longrightarrow S^{D,\dagger}_{4k_0+2}(\mathrm{GL}_2(\mathbb{Z}_5))$$

$$\varphi(x) \longmapsto proj(\varphi)(x) = \varphi\left(x\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\right)\Big|_{\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}} + \sum_{j=0}^{4} \varphi\left(x\begin{pmatrix} 1 & 0 \\ j & 1\end{pmatrix}^{-1}\right)$$

Sketch of the proof

We have the key identity:

$$U_5(\varphi)(x) = i \circ proj(\varphi)(x) - \varphi(x\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}^{-1})|_{\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}}$$

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The matrix for $i \circ proj$ has rank at most d_k^{unr}. For the second part, its upper left $n \times n$ minor has rank:

$$\begin{cases} 0 & \text{if } n \leq d_k^{\mathrm{Iw}}/2 \\ 2n - d_k^{Iw} & \text{otherwise.} \end{cases}$$