The Impossible Vanishing Spectrum

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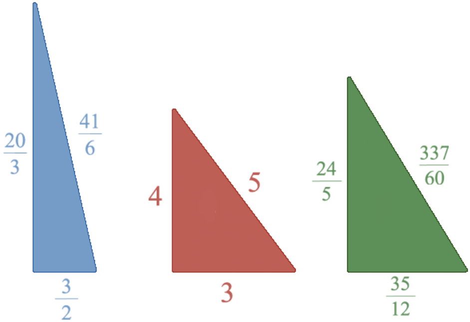

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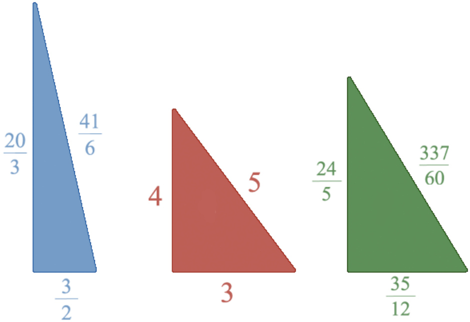

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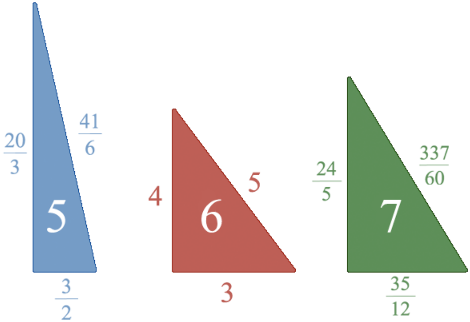

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Theorem

The square-free $t \in \mathbb{N}$ is a congruent number if and only if there exist $m, n \in \mathbb{N}$ such that

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Why?

Let $r_1 : \mathbb{N}_0 \to \{0, 1\}$ be the square indicator function where

$$r_1(n) := \begin{cases} 0 & \text{if } n \text{ is not a square} \\ 1 & \text{if } n = 0 \\ 2 & \text{if } n \text{ is a nonzero square.} \end{cases}$$

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From the previous slide we have that square-free t is congruent if and only if:

$$r_1(m-n)r_1(m)r_1(m+n)r_1(tn) \neq 0$$

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Or alternately, a square-free t is congruent if any only if the double partial sum

$$S_t(X) = \sum_{n,m<X} r_1(m-n)r_1(m)r_1(m+n)r_1(tn)$$

is not the constant zero function.

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Theorem (H., Kuan, Lowry-Duda, Walker[1])

Let $t \in \mathbb{N}$ be squarefree, and let $r_1(n)$ as in the previous slide. Let s be the rank of the elliptic curve $E_t : y^2 = x^3 - t^2 x$ over $\mathbb{Q}$. For $X > 1$, we have the asymptotic expansion:

$$S_t(X) := \sum_{m,n<X} r_1(m+n) r_1(m-n) r_1(m) r_1(tn) = C_t X^{\frac{1}{2}} + O_t((\log X)^{s/2}).$$

in which $C_t := 16 \sum_{h \in \mathcal{H}(t)} \frac{1}{h}$ is the convergent sum over $\mathcal{H}(t)$, the set of hypotenuses, h, of dissimilar primitive right triangles with squarefree part of the area t.

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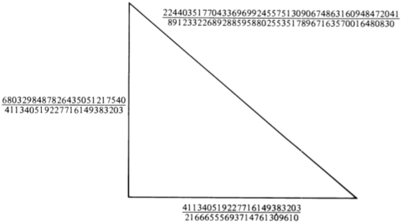

Picture taken from Neal Koblitz's *Introduction to Elliptic Curves and Modular Forms*

So we want to find indirect ways of determining C_t is nonzero.

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Then we would have that when $Q > X$,

$$\sum_{\chi,\psi\ (Q)} S_1(X;\chi,\psi)S_2(X;t,\chi,\psi) = \sum_{m,n<X} r_1(m+n)r_1(m-n)r_1(m)r_1(tn).$$

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When $W(x)$ is a bump function around $x=1$, the above sum counts the number of arithmetic triples of squares where the middle square has size $O(X)$ and one of the other squares has size $O(Y)$.

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To do this, we will take advantage of the automorphic properties of theta functions.

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It is easy to show that $\Gamma_0(N)$ acts on $\mathbb{H}$ by Möbius Maps:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} z = \frac{Az + B}{Cz + D}.$$

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For $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0(4)$, applying Poisson's summation formula on the generators of $\Gamma_0(4)$ allows us to prove that

$$\theta(\gamma z) = \left(\frac{C}{D}\right) \epsilon_D^{-1} \sqrt{Cz + D}\, \theta(z),$$

where $\left(\frac{C}{D}\right)$ denotes Shimura's extension of the Jacobi symbol and $\epsilon_D = 1$ or i depending on if $D \equiv 1$ or 3 (mod 4), respectively.[4]

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It turns out that $\theta(2z)$ is also a holomorphic form of $\Gamma_0(8)$ with nebentypus $\chi(d) := \left(\frac{2}{d}\right)$.

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Let

$$P_h(z, s; \chi) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(8)} \chi(\gamma)\Im(\gamma z)^s e(h\gamma z)$$

denote the level 8, twisted Poincaré series.

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$$\sum_{h=1}^{\infty} \frac{\langle V, P_h(\cdot, \overline{s}; \chi)\rangle}{h^w} = \frac{\Gamma(s-\frac{1}{2})}{(8\pi)^{s-\frac{1}{2}}} \sum_{m=1}^{\infty} \frac{r_1(h)r_1(m)r_1(2m-h)}{m^{s-\frac{1}{2}}h^w}.$$

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From there we wish to take a spectral expansion of $P_h(\cdot, \overline{s}; \chi)$ and rewrite the left-hand side of the above equation as a sum of eigenfunctions and so obtain a meromorphic continuation of the Dirichlet series.

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However we require $V(z)$ to be in $\mathcal{L}^2(\Gamma_0(8), \chi)$ to guarantee this spectral expansion. Thus we have to regularize $V(z)$.

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Let $E(z, s; \chi)$ denote the weight 0, level 8 Eisenstein series with character $\chi := \left(\frac{2}{d}\right)$,

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It turns out that $E(z, \frac{1}{2}; \chi)$ also only has polynomial growth at ∞ and 0, and it matches that of $y^{\frac{1}{2}} \theta(2z)\overline{\theta(z)}$ at each cusp. What remains has exponential decay and so we have that:

$$\widetilde{V}(z) := y^{\frac{1}{2}} \theta(2z)\overline{\theta(z)} - E(z, \tfrac{1}{2}; \chi) \in \mathcal{L}^2(\Gamma_0(8), \chi).$$

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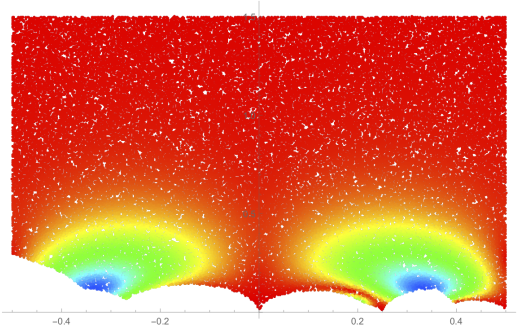

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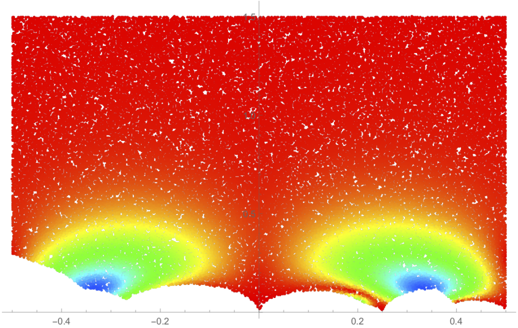

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The above is a heat map of $|\widetilde{V}(z)|$ on the fundamental domain of $\Gamma_0(8)$ in $\mathbb{H}$. Red indicates the value is close to zero, and we notice the function becomes increasingly red as we approach each of the cusps.

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in which $\{\mu_j\}$ denotes an orthonormal basis of Maass cusp forms in $\mathcal{L}^2(\Gamma_0(8), \chi)$, the **discrete spectrum**, and $E_{\mathfrak{a}}(s, z; \chi)$ is the Eisenstein series for level $\Gamma_0(8)$ with character χ for the singular cusp $\mathfrak{a}$, which correspond to the **continuous spectrum**.

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Since $\widetilde{V}(z) \in \mathcal{L}^2(\Gamma_0(8), \chi)$, it has a spectral decomposition.

Since $\Gamma_0(8)$ with $\chi(d) = \left(\frac{2}{d}\right)$ only has two singular cusps, 0 and ∞, the continuous spectrum only has summands arising from those cusps. Furthermore $\langle \widetilde{V}(z), E_{\mathfrak{a}}(\cdot, \frac{1}{2} + it; \chi)\rangle = 0$ for both cusps since the constant term of the Fourier expansion of $\widetilde{V}(z)$ is zero at both cusps.

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So the continuous part of the spectrum appears to vanish.

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So the continuous part of the spectrum appears to vanish.

Furthermore, $\langle E(z, \frac{1}{2}; \chi), \mu_j \rangle = 0$ for all but the constant μ_0 and so the spectral expansion simplifies to

$$\widetilde{V}(z) = \sum_{j \neq 0} \langle V, \mu_j \rangle \mu_j(z) + \langle \widetilde{V}, \mu_0 \rangle \mu_0(z)$$

where we recall that $V(z) = y^{\frac{1}{2}} \theta(2z) \overline{\theta(z)}$.

In a 2016 preprint by Paul Nelson[3] was able to show that $\theta_1\overline{\theta_2}$ will be orthogonal to any cusp form where $\theta_1\overline{\theta_2}$ is the product of unary theta series such as those obtained by imposing congruence conditions in the summation defining $\theta(z)$.

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This makes heuristic sense, if we replace either $\theta(2z)$ or $\theta(z)$ with the residue of the appropriate half-integral weight Eisenstein series. Indeed, we find that unfolding the Eisenstein series before taking the residue produces an analytic symmetric square L-function of μ_j, and so since there is no pole, the residue is zero. Some work would be required to make this rigorous, but it would be expected to push through with a regularization argument.

So we have that

$$\widetilde{V}(z) = \langle \widetilde{V}, \mu_0 \rangle \mu_0(z)$$

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Somewhere we made at least one mistake. Can you find it?

I hope so, because we haven't yet.

Ultimately it is important for us to find the spectral expansion of $\widetilde{V}(z)$ to obtain asymptotic information about

$$H(X,Y) = \sum_{m=1}^{\infty} \sum_{n=-m}^{m} W(\tfrac{m}{X}) W(\tfrac{m-n}{Y}) r_1(m-n) r_1(m) r_1(m+n).$$

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Unfortunately, we can't have confidence in our estimates of $H(X,Y)$ until this contradiction is resolved.

Thanks!

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