Parabolic induction over $\mathbb{Z}_p$

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Problem: Understand (classify?) the irreducible, complex representations of $\mathrm{GL}_n(\mathbb{Z}/p^\ell\mathbb{Z})$.

☐ Limit as $\ell \to \infty$: <u>smooth</u> reps of $\mathrm{GL}_n(\mathbb{Z}_p)$

☐ $\ell = 1$: solved [Frobenius, Schur, Green, Lusztig, ...]

☐ $\ell > 1$, $n = 2, 3$: solved [Kloosterman, Kutzko, Nagornyi, ...]

☐ $\ell > 1$, $n > 3$: open, hard [Hill, Onn, Stasinski, ...]

☐ Classifying irreps of $\mathrm{GL}_n(\mathbb{Z}/p^2\mathbb{Z})$ for all n is <u>wild</u> [Nagornyi]

... So why bother?

- Decompose spaces of automorphic forms [Hecke, Kloosterman, ...]
- Applications to other parts of rep theory [Bushnell-Kutzko, ...]

Problem: Understand (classify?) the irreducible, complex representations of $\mathrm{GL}_n(\mathbb{Z}/p^\ell\mathbb{Z})$.

Strategy 1: induction on ℓ, via Clifford theory $(\mathbb{Z}/p^\ell\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^{\ell-1}\mathbb{Z})$

[Shalika, Kutzko, Hill, Nagornyi, Onn, Stasinski, . . .]

Strategy 2: induction on n, via the “Philosophy of Cusp Forms”

☐ $\ell = 1$: very successful [Green, Harish-Chandra, . . .]

☐ $\ell > 1$: work in progress with E. Meir and U. Onn

☐ we want to be able to use both strategies together

Philosophy of cusp forms for $\mathrm{GL}_n(\mathbb{Z}/p\mathbb{Z})$

Let $G_n = \mathrm{GL}_n(\mathbb{Z}/p\mathbb{Z})$.

For each $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\sum \alpha_i = n$, consider subgroups

$$L_\alpha = \begin{bmatrix} G_{\alpha_1} & & 0 \\ & \ddots & \\ 0 & & G_{\alpha_k} \end{bmatrix}, \quad P_\alpha = \begin{bmatrix} G_{\alpha_1} & & * \\ & \ddots & \\ 0 & & G_{\alpha_k} \end{bmatrix}, \quad U_\alpha = \begin{bmatrix} 1_{\alpha_1} & & * \\ & \ddots & \\ 0 & & 1_{\alpha_k} \end{bmatrix}$$

Parabolic induction:

$$\mathrm{i}_\alpha : \mathrm{Rep}(L_\alpha) \xrightarrow{\text{pull back}} \mathrm{Rep}(P_\alpha) \xrightarrow{\text{induce}} \mathrm{Rep}(G_n)$$

Parabolic restriction:

$$\mathrm{r}_\alpha : \mathrm{Rep}(G_n) \xrightarrow{X \mapsto X^{U_\alpha}} \mathrm{Rep}(L_\alpha)$$

Cusp forms: $X \in \mathrm{Irr}(G_n)$ having $\mathrm{r}_\alpha X = 0$ for all proper α

Philosophy of cusp forms for $\mathrm{GL}_n(\mathbb{Z}/p\mathbb{Z})$

$G_n = \mathrm{GL}_n, \quad n = \sum_{i=1}^k \alpha_i, \quad L_\alpha = \left[\begin{smallmatrix} * & \\ & * \end{smallmatrix}\right], \quad P_\alpha = \left[\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right], \quad U_\alpha = \left[\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right]$

$\mathrm{i}_\alpha : \mathrm{Rep}(L_\alpha) \to \mathrm{Rep}(G_n) \quad \mathrm{r}_\alpha : \mathrm{Rep}(G_n) \to \mathrm{Rep}(L_\alpha)$

cusp forms: $X \in \mathrm{Irr}(G_n)$, $\mathrm{r}_\alpha X = 0$ for all proper α

Theorem: [Green, Harish-Chandra]

☐ Every $X \in \mathrm{Irr}(G_n)$ occurs as a subrep of $\mathrm{i}_\alpha(X_1 \otimes \cdots \otimes X_k)$ for some cusp forms $X_i \in \mathrm{Irr}(G_{\alpha_i})$ (unique up to permutations)

☐ $\mathrm{End}_{G_n}\left[\mathrm{i}_\alpha\left(X_1 \otimes \cdots \otimes X_k\right)\right] \cong \mathbb{C}\left[\begin{matrix}\text{product of} \\ S_m\text{'s}\end{matrix}\right]$

Moral:

$$\boxed{\begin{matrix}\text{Rep theory} \\ \text{of } \mathrm{GL}_n(\mathbb{Z}/p\mathbb{Z})\end{matrix}} = \boxed{\begin{matrix}\text{arithmetic} \\ \text{(classify cuspidals)}\end{matrix}} \bigsqcup \boxed{\begin{matrix}\text{combinatorics} \\ \text{(Young tableaux)}\end{matrix}}$$

Over $\mathbb{Z}/p^\ell\mathbb{Z}$: Harder arithmetic. Same combinatorics?

Philosophy of cusp forms for $\mathrm{GL}_n(\mathbb{Z}/p^\ell\mathbb{Z})$?

$$G_n^\ell = \mathrm{GL}_n^\ell, \quad n = \sum_{i=1}^k \alpha_i, \quad L_\alpha^\ell = \left[\begin{smallmatrix} * & \\ & * \end{smallmatrix}\right], \quad P_\alpha^\ell = \left[\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right], \quad U_\alpha^\ell = \left[\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right]$$

$$\boxed{\text{superscript } \ell \text{ means ``over } \mathbb{Z}/p^\ell\mathbb{Z}\text{'', where } \ell \geq 1}$$

Parabolic induction? $\mathrm{Rep}(L_\alpha^\ell) \xrightarrow{\text{pull back}} \mathrm{Rep}(P_\alpha^\ell) \xrightarrow{\text{induce}} \mathrm{Rep}(G_n^\ell)$ still makes sense ... but the resulting reps are too big.

$$\begin{tikzcd}
\mathrm{Rep}(L_\alpha^\ell) \arrow[r, "\text{pull back to } P_\alpha^\ell \text{ then induce}"] \arrow[d, "\text{pull back}"'] \arrow[dr, phantom, "\text{doesn't commute!}"] & \mathrm{Rep}(G_n^\ell) \arrow[d, "\text{pull back}"] \\
\mathrm{Rep}(L_\alpha^{\ell+1}) \arrow[r, "\text{pull back to } P_\alpha^{\ell+1} \text{ then induce}"'] & \mathrm{Rep}(G_n^{\ell+1})
\end{tikzcd}$$

Proposal for "parabolic induction" over $\mathbb{Z}/p^\ell\mathbb{Z}$

$$G_n^\ell = \mathrm{GL}_n^\ell, \quad n = \sum_{i=1}^k \alpha_i, \quad L_\alpha^\ell = \left[\begin{smallmatrix} * & \\ & * \end{smallmatrix}\right], \quad P_\alpha^\ell = \left[\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right], \quad U_\alpha^\ell = \left[\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right]$$

Parabolic induction? [CMO, cf. Dat] $\mathrm{i}_\alpha^\ell : \mathrm{Rep}(L_\alpha^\ell) \to \mathrm{Rep}(G_n^\ell)$

$$\mathrm{i}_\alpha^\ell X := \mathrm{Image}\left[\mathrm{ind}_{(P_\alpha^\ell)^{\mathrm{t}}}^{G_n^\ell} X \xrightarrow[\text{standard intertwiner}]{\int_{U_\alpha^\ell}} \mathrm{ind}_{P_\alpha^\ell}^{G_n^\ell} X\right]$$

☐ For $\ell = 1$, new $\mathrm{i}_\alpha^1 =$ old i_α [Howlett-Lehrer]

☐ i_α^ℓ is compatible with Clifford theory upon changing ℓ: e.g.,

$$\begin{tikzcd}
\mathrm{Rep}(L_\alpha^\ell) \arrow[r, "\mathrm{i}_\alpha^\ell"] \arrow[d, "\text{pull back}"'] \arrow[dr, phantom, "\text{commutes}"] & \mathrm{Rep}(G_n^\ell) \arrow[d, "\text{pull back}"] \\
\mathrm{Rep}(L_\alpha^{\ell+1}) \arrow[r, "\mathrm{i}_\alpha^{\ell+1}"'] & \mathrm{Rep}(G_n^{\ell+1})
\end{tikzcd}$$

☐ $\exists$ an adjoint restriction functor r_α^ℓ, thus a notion of cusp forms.

Philosophy of cusp forms for $\mathrm{GL}_n(\mathbb{Z}/p^\ell\mathbb{Z})$?

Conjecture: (analogue of Green's theorem for all $\ell \geq 1$)

☐ Every $X \in \mathrm{Irr}(G_n^\ell)$ occurs as a subrep of $\mathrm{i}_\alpha^\ell(X_1 \otimes \cdots \otimes X_k)$ for some cusp forms $X_i \in \mathrm{Irr}(G_{\alpha_i}^\ell)$ (unique up to permutations)

☐ $\mathrm{End}_{G_n^\ell}\left[\mathrm{i}_\alpha^\ell\left(X_1 \otimes \cdots \otimes X_k\right)\right] \cong \mathbb{C}\left[\begin{array}{c}\text{product of}\\ S_m\text{'s}\end{array}\right]$

Theorem: It's enough to verify the conjecture for nilpotent representations (with $\mathbb{Z}_p$ replaced by a general ring of integers).

(nilpotence: Clifford-theoretic condition involving restriction to the minimal congruence subgroup, $\ker(G_n^\ell \twoheadrightarrow G_n^{\ell-1}) \cong M_n(\mathbb{Z}/p\mathbb{Z})$)

Theorem: For $\alpha = (1, \ldots, 1)$:

$$\mathrm{End}_{G_n}\left[\mathrm{i}_\alpha^\ell\left(X_1 \otimes \cdots \otimes X_n\right)\right] \cong \mathbb{C}\left[\begin{array}{c}\text{product of}\\ S_m\text{'s}\end{array}\right]$$

Coda: equivariant homology of Bruhat-Tits buildings

G: p-adic reductive group (e.g., $\mathrm{GL}_n(\mathbb{Q}_p)$)

Theorem: [Higson-Nistor, Schneider, Bernstein, Keller]

$$\boxed{\begin{array}{c}\mathrm{H}_*^G(\mathrm{BT}(G)) : \text{equivariant} \\ \text{homology of Bruhat-Tits} \\ \text{building} \approx \textit{geometry} + \\ \textit{rep thy of cmpct sbgrps}\end{array}} \cong \boxed{\begin{array}{c}\mathrm{HP}_*(\mathrm{Rep}(G)) : \text{periodic} \\ \text{cyclic homology of } \mathrm{Rep}(G) \\ \approx \textit{cohomology of Irr(G)}\end{array}}$$

Question: how does parabolic induction fit into this picture?

Theorem: For $G = \mathrm{SL}_2$, $L = \begin{bmatrix} * & \\ & * \end{bmatrix}$ (and perhaps more generally):

$$\begin{tikzcd}
\mathrm{H}_*^L(\mathrm{BT}(L)) \arrow[r, "\cong"] \arrow[d, swap, "\boxed{\text{assemblage of } \mathrm{i}_\alpha^\ell\text{s}}"] & \mathrm{HP}_*(\mathrm{Rep}(L)) \arrow[d, "\text{parabolic induction}"] \\
\mathrm{H}_*^G(\mathrm{BT}(G)) \arrow[r, "\cong"] & \mathrm{HP}_*(\mathrm{Rep}(G))
\end{tikzcd}$$