The Explicit Sato-Tate Conjecture in Arithmetic Progressions

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Motivation

Theorem (Prime Number Theorem)

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Theorem

Refinement to arithmetic progressions: Let a, q *be such that* $\gcd(a, q) = 1$*. Then*

$$\pi(x; q, a) := \#\{p \leq x : p \textit{ prime and } p \equiv a \bmod q\} \sim \frac{1}{\varphi(q)}\mathrm{Li}(x).$$

Modular Forms

- Recall that a modular form of weight k on $SL_2(\mathbb{Z})$ is a function $f : \mathbb{H} \to \mathbb{C}$ with

$$f(z) = \sum_{n=0}^{\infty} a_f(n) q^n, \;\; q = e^{2\pi i z}$$

and

$$f(\gamma z) = (cz + d)^k f(z) \text{ for all } \gamma \in SL_2(\mathbb{Z}).$$

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- By restricting to the action of a congruence subgroup $\Gamma \subset SL_2(\mathbb{Z})$ of level N, we can associate that level to our modular form $f(z)$.
- We say a modular form is a cusp form if it vanishes at the cusps of Γ; hence $a_f(0) = 0$ for a cusp form $f(z)$.

Newforms

- We say f is a Hecke eigenform if it is a cusp form and

$$T_n f = \lambda(n) f \text{ for } n = 1, 2, 3, \ldots,$$

where T_n is the Hecke operator.

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- We consider holomorphic cuspidal newforms of even weight $k \geq 2$ and squarefree level N.

The Ramanujan Tau Function

- Ramanujan tau function:

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Conjecture (Lehmer)

For all $n \geq 1, \tau(n) \neq 0$.

The Sato-Tate Law

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- Natural question: What is the distribution of the sequence $\{\theta_p\}$?

The Sato-Tate Law (Continued)

Theorem(Barnet-Lamb, Geraghty, Harris, Taylor)

Let $f(z) \in S_k^{new}(\Gamma_0(N))$ be a non-CM newform. If $F : [0, \pi] \to \mathbb{C}$ is a continuous function, then

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leq x} F(\theta_p) = \int_0^{\pi} F(\theta) d\mu_{ST}$$

where $d\mu_{ST} = \frac{2}{\pi}\sin^2(\theta)d\theta$ is the Sato-Tate measure. Further

$$\pi_{f,I}(x) := \#\{p \leq x : \theta_p \in I\} \sim \mu_{ST}(I)\mathrm{Li}(x).$$

Symmetric Power L-functions

- We begin by writing

$$f(z) = \sum_{m=1}^{\infty} a_f(m) q^m = \sum_{m=1}^{\infty} m^{\frac{k-1}{2}} \lambda_f(m) q^m.$$

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- From this normalization, we have

$$L(s,f) = \prod_p \left(1 - e^{i\theta_p} p^{-s}\right)^{-1} \left(1 - e^{-i\theta_p} p^{-s}\right)^{-1},$$

and the n-th symmetric power L-function

$$L\left(s, \mathrm{Sym}^n f\right) = \left(\prod_{p \nmid N} \prod_{j=0}^{n} \left(1 - e^{ij\theta_p} e^{(j-n)i\theta_p} p^{-s}\right)^{-1} \right) \left(\prod_{p \mid N} L_p(s)^{-1} \right)$$

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- To pass to arithmetic progressions, we consider $L(s,\mathrm{Sym}^n f\otimes\chi)$.

Previous Work

- Define $\pi_{f,I}(x) = \#\{p \leq x : \theta_p \in I\}$ and let $\mu_{ST}(I)$ denote the Sato-Tate measure of a subinterval $I \subset [0, \pi]$.

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- Rouse and Thorner (2017): under certain analytic hypotheses on the symmetric power L-functions,

$$|\pi_{f,I}(x) - \mu_{ST}(I)Li(x)| \leq 3.33x^{3/4} - \frac{3x^{3/4}\log\log x}{\log x} + \frac{202x^{3/4}\log \mathfrak{q}(f)}{\log x}$$

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- Rouse-Thorner also leads to an explicit upper bound for the Lang-Trotter conjecture, which predicts the asymptotic of the number of primes for which $a_f(p) = c$ for a fixed constant c.

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  - The Generalized Riemann Hypothesis for the twisted symmetric power L-functions $L(s, \mathrm{Sym}^n f \otimes \chi)$.
  - The existence of an analytic continuation of $L(s, \mathrm{Sym}^n f \otimes \chi)$ to an entire function on $\mathbb{C}$ (and a corresponding functional equation).
  - Assumptions about the form of the above completed L-function, including its gamma factor and conductor.

Our Results

Assuming the aforementioned hypotheses, we prove:

Sato-Tate Conjecture for Primes in Arithmetic Progressions

Fix a modulus q. Let $\phi(t)$ be a compactly supported C^∞ test function, and set $\phi_x(t) = \phi(t/x)$. For $x \geq \max\{3.5 \times 10^7, 7400(q \log q)^2\}$:

$$\left| \sum_{\substack{p \nmid N,\, \theta_p \in I \\ p \equiv a(q)}} \log(p)\phi_x(p) - \frac{x\mu_{ST}(I)}{\varphi(q)} \left(\int_{-\infty}^{\infty} \phi(t)dt \right) \right| \leq \frac{Cx^{3/4}\sqrt{\log x}}{\sqrt{\varphi(q)}}$$

for some computable constant C depending on ϕ.

Our Results (continued)

Theorem

Let $\tau(n)$ *be the Ramanujan tau function. Then for* $x \geq 10^{50}$,

$$\#\{x < p \leq 2x \mid \tau(p) = 0\} \leq 5.973 \times 10^{-7} \frac{x^{3/4}}{\sqrt{\log x}}.$$

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As a consequence, we obtain the following strong evidence in favor of Lehmer's conjecture:

Theorem

Let $\tau(n)$ be the Ramanujan tau function. Then

$$\lim_{X \to \infty} \frac{\#\{n \leq X \mid \tau(n) \neq 0\}}{X} > 1 - 5.2 \times 10^{-14}.$$

Proof Outline: Bounding $\#\{x < p \leq 2x \mid \tau(p) = 0\}$

- If $\tau(p) = 0$, then $\theta_p = \pi/2$ and, by the work of Serre (1981), p is in one of 33 possible residue classes modulo

$$q = 24 \times 49 \times 3094972416000.$$

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$$\frac{33}{\log x} \sum_{\substack{p \\ \theta_p = \pi/2 \\ p \equiv a(q)}} \log(p)\phi_x(p) \geq \#\{x < p \leq 2x \mid \tau(p) = 0\}.$$

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Bounding the $\theta_p \in [\pi/2, \pi/2]$ condition

Rouse-Thorner (2017) construct trigonometric polynomials

$$F^{\pm}_{I,M}(\theta) = \sum_{n=0}^{M} \hat{F}^{\pm}_{I,M}(n) U_n(\cos\theta)$$

which satisfy $\forall x \in [0, \pi]$,

$$F^{-}_{I,M}(x) \leq \chi_I(x) \leq F^{+}_{I,M}(x)$$

and best approximate the indicator function for any interval $I \in [0, \pi]$. Using these we can expand out the sum from the previous slide.

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Fourier Expansion

Therefore, setting $I = [\pi/2 - \epsilon, \pi/2 + \epsilon]$:

$$\sum_{\substack{p \\ \theta_p = \pi/2 \\ p \equiv a(q)}} \frac{\log p}{\log x} \phi_x(p)$$

$$\leq \frac{1}{\log x} \sum_{n=0}^{M} |\hat{F}^{+}_{I,M}(n)| \frac{1}{\varphi(q)} \sum_{\chi(q)} \overline{\chi}(a) \left| \sum_{p} U_n(\cos\theta_p) \log(p) \chi(p) \phi_x(p) \right| .$$

Through contour integration we can bound this innermost sum, and consequently, obtain a bound for the entire expression.

Proof Outline: The Contour Integral

The innermost sum is related to the contour integral of the n-th symmetric L-function twisted by χ:

$$\sum_{p^j} U_n(\cos(j\theta_p))\chi(p^j)\log(p) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} -\frac{L'}{L}(s, \mathrm{Sym}^n f \otimes \chi)\Phi_x(s)\, ds.$$

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By pushing this contour to $-\infty$ and summing the residues from the zeros of $L(s, \mathrm{Sym}^n f \otimes \chi)$, we have

$$\sum_p U_n(\cos\theta_p)\log(p)\chi(p)\phi_x(p) = \delta_{\substack{n=0\\ \chi=\chi_0}} \Phi(1)x - \sum_\rho \Phi(\rho)x^\rho + O(n\sqrt{x}).$$

Proof Outline: From the Contour Integral to the Final Bound

Evaluates to

$$\left|\sum_{p} U_n(\cos\theta_p)\log(p)\chi(p)\phi_x(p)\right| \le \delta_{\substack{n=0\\ \chi=\chi_0}} \Phi(1)x + O(n\log n\sqrt{x})$$

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where we can compute explicit bounds for the error term. Then,

$$\sum_{\substack{p \\ \theta_p=\pi/2 \\ p\equiv a(q)}} \frac{\log p}{\log x}\phi_x(p) \leq \frac{1}{\log x}\left(\frac{1.33x}{\varphi(q)M} + 7.63M\log M\sqrt{x} + O(M\sqrt{x})\right).$$

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Selecting $M = 6.894\times 10^{-9}\frac{x^{1/4}}{\sqrt{\log x}}$, gives us our final bound. $\square$

References

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