A Dirichlet Series for the Congruent Number Problem

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Joint work with Chan Ieong Kuan, David Lowry-Duda and Alexander Walker

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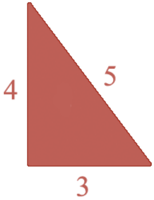

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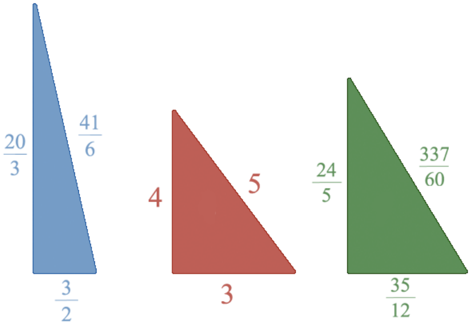

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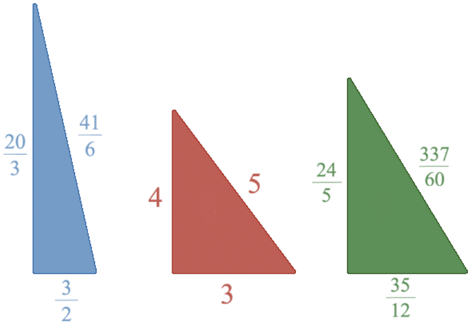

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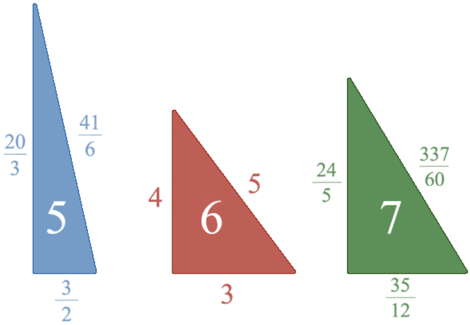

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There is a known (awful) way to find out if n is a congruent number, if it is indeed a congruent number. We can take the parametrization of primitive pythagorean triples due to Euclid and then wait until our congruent number shows up.

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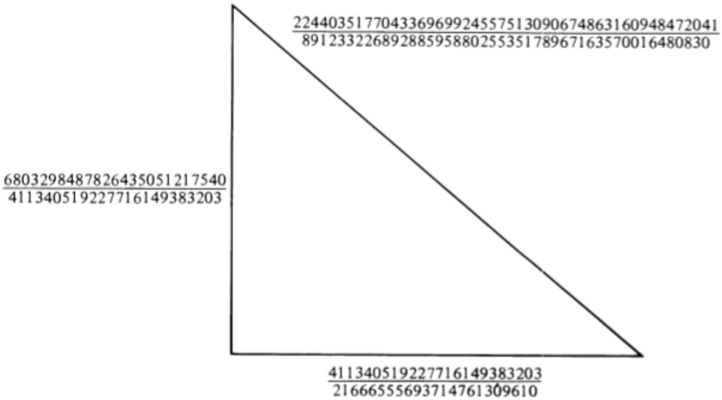

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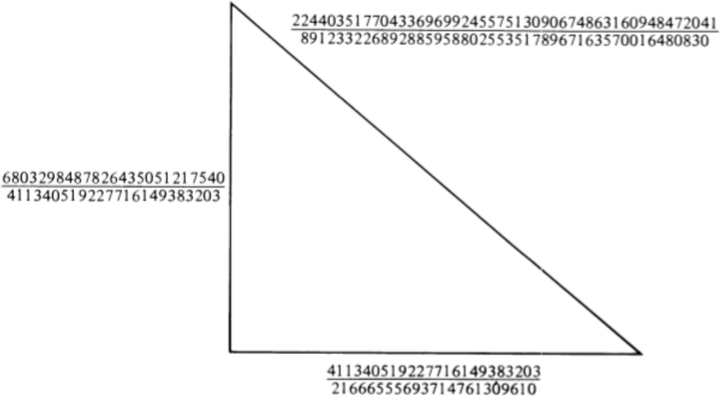

This is the simplest rational right triangle with area 157, discovered by Don Zagier around 1990.[2]

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Observation

The square-free integer t is a congruent number if and only if it is the square-free part of $xy/2$ where $(x, y, z) \in \mathbb{Z}^3$ is a primitive Pythagorean triple, that is $x^2 + y^2 = z^2$ and x, y, z are pairwise coprime.

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Theorem (Tunnell, 1983[2])

Let t be an odd square-free natural number. Consider the two conditions:

(A) *t is congruent;*

(B) *the number of triples of integers (x, y, z) satisfying $2x^2 + y^2 + 8z^2 = t$ is equal to twice the number of triples satisfying $2x^2 + y^2 + 32z^2 = t$.*

Then (A) implies (B); and, if a weak form of the Birch-Swinnerton-Dyer conjecture is true, then (B) also implies (A).

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We see that if (B) does imply (A), then all we have to do is perform a search that would take a computable, finite amount of time that depends on the size of t. This would be an acceptable solution to the Congruent Number Problem.

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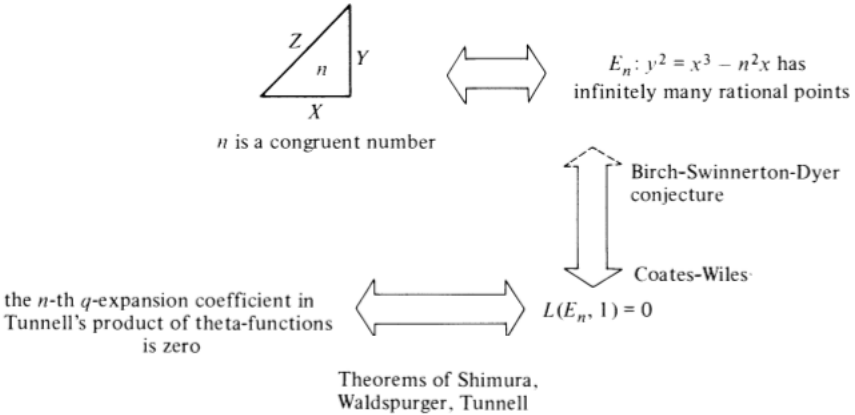
Here's a helpful diagram made by Koblitz[2]:



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By scaling, we see equivalently that square-free t is a congruent number if and only if there are natural numbers $m, n \in \mathbb{N}$ such that $(m - tn), m, (m + tn)$, and n are all squares.

Proof of equivalence:

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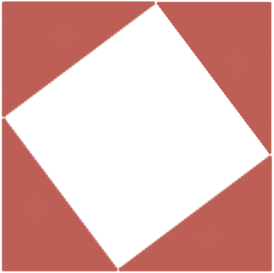
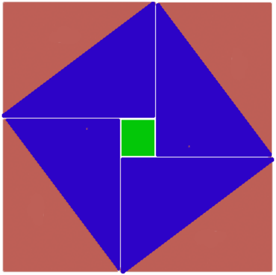
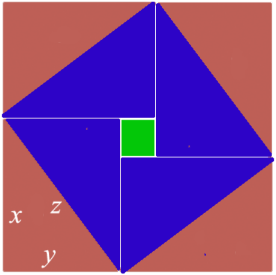


Proof of equivalence: And we can make a smaller square by doing this.



Proof of equivalence: Now we label the relevant quantities.



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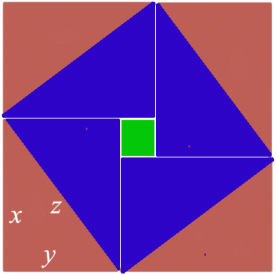

By our hypothesis, $x, y, z \in \mathbb{Q}$. We see that each of the above squares are rational, with side lengths $(x + y), z$ and $(x - y)$ in descending order, and that the difference in area between subsequent squares are the triangles, which we can say have area $n \in \mathbb{N}$.

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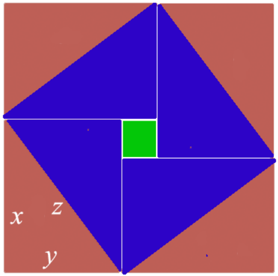

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Multiplying by a factor of $\frac{1}{2}$ we see $(z/2)^2 \pm n$ are both squares. This argument is reversible.

Shifted Sums

Let $s : \mathbb{N} \to \{0, 1\}$ be the square indicator function where

$$s(n) := \begin{cases} 0 & \text{if } n \text{ is not a square} \\ 1 & \text{if } n \text{ is a square} \end{cases}$$

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With this we can reformulate the "arithmetic progression of squares" argument to be that square-free t is congruent if and only if:

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Or alternately, a square-free t is congruent if any only if the double partial sum

$$S_t(X) = \sum_{n,m<X} s(m-n)s(m)s(m+n)s(tn)$$

is not the constant zero function.

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Theorem (H., Kuan, Lowry-Duda, Walker)

Let $t \in \mathbb{N}$ be squarefree, and let $s(n)$ be the square-indicator function. Let r be the rank of the elliptic curve $E_t : y^2 = x^3 - t^2 x$ over $\mathbb{Q}$. For $X > 1$, we have the asymptotic expansion:

$$S_t(X) := \sum_{m,n<X} s(m+n)s(m-n)s(m)s(tn) = C_t X^{\frac{1}{2}} + O_t((\log X)^{r/2}).$$

in which $C_t := \sum_{h \in \mathcal{H}(t)} \frac{1}{h}$ is the convergent sum over $\mathcal{H}(t)$, the set of hypotenuses, h, of dissimilar primitive right triangles with squarefree part of the area t.

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We use the correspondence between rational right triangles and rational points on $E_t(\mathbb{Q})$, along with the known asymptotics of the number of rational points of bounded height, to get the asymptotic equivalence.

We note that $C_t \neq 0$, and equivalently $S_t(X)$ has polynomial growth in X, if and only t is a congruent number. Unfortunately, computing this series for large X by taking partial sums is no more efficient than running Euclid's ineffective algorithm and waiting for t to appear in it.

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The hope is that taking inverse Mellin transforms of shifted multiple Dirichlet series of automorphic forms will afford us an alternate avenue for computing C_t directly.

Theta Functions

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Suppose for $z \in \mathbb{H}$ we define the **theta function**:

$$\theta(z) := \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z} = \sum_{n=0}^{\infty} r_1(n) e^{2\pi i n z} = 1 + 2\sum_{n=1}^{\infty} s(n) e^{2\pi i n z}$$

which is uniformly convergent on compact subsets of $\mathbb{H}$.

For $N \in \mathbb{N}$, let $\Gamma_0(N)$ denote the congruence subgroup:

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For $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right) \in \Gamma_0(4)$, applying Poisson's summation formula on the generators of $\Gamma_0(4)$ allows us to prove that

$$\theta\left(\gamma z\right) = \left(\tfrac{C}{D}\right) \epsilon_D^{-1} \sqrt{Cz+D}\, \theta(z),$$

where $\left(\frac{C}{D}\right)$ denotes Shimura's extension of the Jacobi symbol and $\epsilon_D = 1$ or i depending on if $D \equiv 1$ or 3 (mod 4), respectively.[3]

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We refer to $\theta(z)$ as a weight $1/2$ **holomorphic form** on $\Gamma_0(4)$.

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Then the orthogonality properties for Dirichlet characters give us that

$$\sum_{\substack{\chi,\psi \\ (Q)}} S^{-}_{\chi,\psi}(X) S^{+}_{\chi,\psi}(X;t) = \sum_{m,n<X} r_1(m-n) r_1(m) r_1(m+n) r_1(tn).$$

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Indeed, we can define the **twisted** theta function

$$\theta_\chi(z) = \sum_{m=0}^{\infty} r_1(m)\chi(m)e^{2\pi imz}$$

which, for $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0(4Q^2)$, satisfies

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The take-away is that $r_1(m)$, $r_1(m)\chi(m)$ and $r_1(tn)\chi(n)$ are all the Fourier coefficients of holomorphic forms, and so we know how to get the meromorphic continuations of the shifted multiple Dirichlet series,

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to all $(s,w)\in\mathbb{C}^2$ by using spectral expansions of Poincaré series.

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to all $(s,w)\in\mathbb{C}^2$ by using spectral expansions of Poincaré series.

By taking inverse Mellin transforms of these objects, we are able to get the asymptotics of $S^{-}_{\chi,\psi}(X)$ and $S^{+}_{\chi,\psi}(X;t)$ and by averaging those over characters we may get the desired asymptotics of $S_t(X)$.

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- It is not obvious that when this is done it will give us a coherent asymptotic formula for $S_t(X)$ or if the Q term will dominate and obscure the relevant asymptotic information.

- This approach could yield an asymptotic formula that is no better than the one we have already proven, and so not provide a reasonable alternative algorithm to checking if t is a congruent number.

The Congruent Number Zeta Function

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$$Z_t(s) := \sum_{m,n=1}^{\infty} \frac{s(m-n)s(m)s(m+n)s(tn)}{m^s}.$$

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From the reasoning in the proof of the asymptotic formula for $S_t(X)$, it is not difficult to show that

$$Z_t(S) = \zeta(2s) \sum_{h \in H(t)} \frac{1}{h^{2s}}$$

where, again, $\mathcal{H}(t)$, the set of hypotenuses, h, of dissimilar primitive right triangles with squarefree part of the area t.

So we can think of $L_t(s)$

$$Z_t(S) = \sum_{m,n=1}^{\infty} \frac{s(m-n)s(m)s(m+n)s(tn)}{m^s} = \zeta(2s) \sum_{h \in H(t)} \frac{1}{h^{2s}}$$

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$$S_t(X) := \sum_{m,n<X} s(m+n)s(m-n)s(m)s(tn) = C_t X^{\frac{1}{2}} + O_t((\log X)^{r/2}).$$

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The hope is that we can use spectral methods to represent $Z_t(s)$ in terms of spectral expansions and so bypass shifted sums altogether. It is also not obvious how whether this will work.

Thanks!

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