On Thue equations

(Joint results with Michel Waldschmidt)

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1. Introduction

Some infinite families of diophantine Thue equations having only trivial solutions (or a finite number of integral solutions) have been exhibited by a few mathematicians:

Thomas, Győry, Schlickewei, Pethő, Evertse, Gaál, Tichy, Heuberger, de Weger, Fuchs, Lettl, Voutier, Chen, Mignotte, Tzanakis, Wakabayashi, Togbé, Ziegler, Berczes, Walsh, Halter-Koch, Dujella, etc. ... and of course Michel Waldschmidt.

2. The first family of Thomas

In 1990 Thomas studied the families of diophantine equations

$$F_n(X, Y) = c$$

where

$$F_n(X, Y) = X^3 - (n-1)X^2Y - (n+2)XY^2 - Y^3 \quad \textit{and} \quad c = \pm 1.$$

The polynomial $F_n(X, Y)$ is the homogenized form of the minimal polynomial $f_n(X)$ of Shanks's simplest cubic fields, namely

$$f_n(X) = X^3 - (n-1)X^2 - (n+2)X - 1.$$

Theorem (Thomas). *Let*

$$F_n(X,Y) = X^3 - (n-1)X^2Y - (n+2)XY^2 - Y^3 \quad \textit{with} \quad c = \pm 1.$$

(i) *For $n \geq 1.365 \times 10^7$, there are only the trivial solutions:*

$$(c,0), \quad (0,c), \quad (c,-c).$$

(ii) *For $0 \leq n \leq 1000$, the other solutions are:*

$$\begin{array}{lcllll} n=0 & : & (-9c,5c), & (-c,2c), & (2c-c), \\ & & (4c,-9c), & (5c,4c), & (-c,-c) \\ n=1 & : & (-3c,2c), & (c,-3c), & (2c,c); \\ n=3 & : & (-7c,-2c), & (-2c,9c), & (9c,-7c). \end{array}$$

Theorem (Mignotte). *For $n \geq 0$, the only solutions are the above ones.*

3. Our main theorem

Theorem (Waldschmidt-L).
Let K be an algebraic number field of degree $d \geq 3$. For every unit ε of degree at least 3 in K, except for a finite number of them, the following holds true: Let $f_\varepsilon(X)$ be the minimal polynomial of ε and let $F_\varepsilon(X, Y)$ be the homogenized binary form associated to $f_\varepsilon(X)$. Then the solutions (x, y) of the Thue equation

$$F_\varepsilon(X, Y) = 1$$

are given by $xy = 0$.

The proof (which is not effective) uses the subspace lemma of Wolfgang Schmidt.

4. A general result involving powers of units

Let $d \geq 3$ be a given integer. Let $F(X, Y)$ be a monic irreducible binary form in $\mathbf{Z}[X, Y]$ satisfying $F(0, 1) = \pm 1$ and that we write as

$$F(X, Y) = \prod_{j=1}^{d} (X - \alpha_j Y)$$

with $|\alpha_1| \leq |\alpha_2| \leq \cdots \leq |\alpha_d|$. Denote by R the regulator of the number field $\mathbf{Q}(\alpha_1)$. Further, let $\lambda = |\alpha_d|$. For $a \geq 0$, consider the polynomial of $\mathbf{Z}[X, Y]$ defined by

$$F_a(X, Y) = \prod_{j=1}^{d} (X - \alpha_j^a Y).$$

Theorem (Waldschmidt-L). *Assume that α_1 is not a root of unity. There exists an effectively computable constant κ, depending only on d, with the following property. Let $(x, y, a) \in \mathbf{Z}^3$ satisfy*

$$xy \neq 0, \quad [\mathbf{Q}(\alpha_1^a) : \mathbf{Q}] = d, \quad F_a(x, y) = \pm 1.$$

Then,

$$|a| \leq \kappa \lambda^{d^4} R \, log \, R.$$

Moreover, there exists another effectively computable constant κ such that

$$max\{\log|x|, \ \log|y|\} < \kappa R \log(R)\big(R + |a| \log(\lambda)\big).$$

Our proof actually gives a much stronger estimate which depends on the following parameter $\mu > 1$ defined by

$$\mu = \begin{cases} \max\{2, \lambda\} & \text{if } |\alpha_1| = |\alpha_{d-1}| \text{ or } |\alpha_2| = |\alpha_d|, \\ \min\left\{\dfrac{|\alpha_{d-1}|}{|\alpha_1|}, \dfrac{|\alpha_d|}{|\alpha_2|}\right\} & \text{if } |\alpha_1| < |\alpha_2| = |\alpha_{d-1}| < |\alpha_d|, \\ \dfrac{|\alpha_{d-1}|}{|\alpha_2|} & \text{if } |\alpha_2| < |\alpha_{d-1}|. \end{cases}$$

Theorem. *There exists an effectively computable constant κ such that*

$$|a| \leq \kappa R \, \frac{\log \lambda}{\log \mu} \, (R + \log \lambda) \log(R + \log \lambda).$$

Thank you very much!