Determining Hilbert Modular Forms by Central Values of Rankin-Selberg Convolutions

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Motivation

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Theorem (Luo-Ramakrishnan, 1997)

Let $l \equiv l' \equiv 0 \mod 2$, and let g and g' be normalized eigenforms in $\mathcal{S}_l^{new}(N)$ and $\mathcal{S}_{l'}^{new}(N')$, respectively. Suppose that

$$L\left(g \otimes \chi_d, \frac{1}{2}\right) = L\left(g' \otimes \chi_d, \frac{1}{2}\right)$$

for almost all primitive quadratic characters χ_d of conductor prime to NN'. Then $g = g'$.

Motivation - Level Aspect on $\mathrm{GL}_2(\mathbb{Q})$

Theorem (Luo, 1999)

Let $l \equiv l' \equiv k \equiv 0 \mod 2$, and let g and g' be normalized eigenforms in $\mathcal{S}_l^{new}(N)$ and $\mathcal{S}_{l'}^{new}(N')$, respectively. If there exist infinitely many primes p such that

$$L\left(f \otimes g, \frac{1}{2}\right) = L\left(f \otimes g', \frac{1}{2}\right)$$

for all normalized newforms f in $S_k^{new}(p)$, then we have $g = g'$.

Motivation - Weight Aspect on $\mathrm{GL}_2(\mathbb{Q})$

Theorem (Ganguly-Hoffstein-Sengupta, 2009)

Let $l \equiv l' \equiv k \equiv 0 \mod 2$, and let g and g' be normalized eigenforms in $\mathcal{S}_l(1)$ and $\mathcal{S}_{l'}(1)$, respectively. If

$$L\left(f \otimes g, \frac{1}{2}\right) = L\left(f \otimes g', \frac{1}{2}\right)$$

for all normalized eigenforms $f \in S_k(1)$ for infinitely many k, then $g = g'$.

Motivation - Weight Aspect on $\mathrm{GL}_2(\mathbb{Q})$

Theorem (Ganguly-Hoffstein-Sengupta, 2009)

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- (Zhang, 2011) $g \in \mathcal{S}_l^{\text{new}}(\mathfrak{n})$ and $g' \in \mathcal{S}_{l'}^{\text{new}}(\mathfrak{n}')$, $(f \in \mathcal{S}_k(1))$.

Question

Can one generalize those results to Hilbert modular forms?

Setting

- F: totally real number field of degree n
- $\mathcal{O}_F$: ring of integers in F
- $\mathcal{D}_F$: different ideal of F
- h^+: the narrow class number
    - $\{\overline{\mathfrak{a}}\}$: a set of representatives of the narrow class group
- embeddings of F: $\{\sigma_1, \cdots, \sigma_n\}$.
    - For $x \in F$ and $j \in \{1, \ldots, n\}$, we set $x_j = \sigma_j(x)$
    - $x \gg 0$ if $x_j > 0 \; \forall \; j$

Hilbert Modular Form

- $\mathbf{f} := (f_1, \ldots, f_{h^+})$ with $f_i \in \mathcal{S}_k(\Gamma_{\bar{\mathfrak{a}}_i}(\mathfrak{n}))$.
  - $f_i \,:\, \mathfrak{h}^n \to \mathbb{C}$
  - $f_i|_k\gamma = f_i$ for all $\gamma \in \Gamma_{\bar{\mathfrak{a}}_i}(\mathfrak{n}))$
- Fourier coefficients at $\mathfrak{m} \subset \mathcal{O}_F$: $C_{\mathbf{f}}(\mathfrak{m})$
- $k = (k_1, \ldots, k_n)$ with $k_1 \equiv \cdots \equiv k_n \equiv 0 \mod 2$
- $\mathbf{f}$ is primitive $\Leftrightarrow$ $\mathbf{f}$ is a normalized eigenform in $\mathcal{S}_k^{\text{new}}(\mathfrak{n})$.
- $\Pi_k(\mathfrak{n})$: a set of all primitive forms of weight k and level $\mathfrak{n}$.
- Rankin-Selberg convolution of $\mathbf{f} \in \Pi_k(\mathfrak{q})$ and $\mathbf{g} \in \Pi_l(\mathfrak{n})$ is defined as

$$L(\mathbf{f} \otimes \mathbf{g}, s) = \zeta_F^{\mathfrak{nq}}(2s) \sum_{\mathfrak{m} \subset \mathcal{O}_F} \frac{C_{\mathbf{f}}(\mathfrak{m}) C_{\mathbf{g}}(\mathfrak{m})}{\mathrm{N}(\mathfrak{m})^s}$$

Main Theorem I (Level Aspect)

Theorem (Hamieh, T.)

Let $\mathbf{g} \in \Pi_l(\mathfrak{n})$ and $\mathbf{g}' \in \Pi_{l'}(\mathfrak{n}')$, with the weights l and l' being in $2\mathbb{N}^n$. Let $k \in 2\mathbb{N}^n$ be fixed, and suppose that there exist infinitely many prime ideals $\mathfrak{q}$ such that

$$L\left(\mathbf{f} \otimes \mathbf{g}, \frac{1}{2}\right) = L\left(\mathbf{f} \otimes \mathbf{g}', \frac{1}{2}\right)$$

for all $\mathbf{f} \in \Pi_k(\mathfrak{q})$. Then $\mathbf{g} = \mathbf{g}'$.

Main Theorem II (Weight Aspect)

Theorem (Hamieh, T.)

Let $\mathbf{g} \in \Pi_l(\mathfrak{n})$ *and* $\mathbf{g}' \in \Pi_{l'}(\mathfrak{n}')$, *with the weights* l *and* l' *being in* $2\mathbb{N}^n$. *Let* $\mathfrak{q}$ *be a fixed prime ideal. If*

$$L\left(\mathbf{f} \otimes \mathbf{g}, \frac{1}{2}\right) = L\left(\mathbf{f} \otimes \mathbf{g}', \frac{1}{2}\right)$$

for all $\mathbf{f} \in \Pi_k(\mathfrak{q})$ *for infinitely many* $k \in 2\mathbb{N}^n$, *then* $\mathbf{g} = \mathbf{g}'$.

Main Idea

Let $\omega_{\mathbf{f}} = \dfrac{(4\pi)^{k-1}|d_F|^{1/2}\langle \mathbf{f}, \mathbf{f}\rangle_{\mathfrak{q}}}{\Gamma(k-1)}$.

For any prime ideal $\mathfrak{p}$ of F (away from bad primes), we consider a twisted first moment,

$$\sum_{\mathbf{f}\in\Pi_k(\mathfrak{q})} L\left(\mathbf{f}\otimes\mathbf{g}, \frac{1}{2}\right) C_{\mathbf{f}}(\mathfrak{p})\omega_{\mathbf{f}}^{-1}$$

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asymptotically. Further analyzation of the above expression will allow us to conclude that if

$$L\left(\mathbf{f}\otimes\mathbf{g},\frac{1}{2}\right) = L\left(\mathbf{f}\otimes\mathbf{g}',\frac{1}{2}\right)$$

for all $\mathbf{f}\in\Pi_k(\mathfrak{q})$ (for infinitely many k or $\mathfrak{q}$), then $C_{\mathbf{g}}(\mathfrak{p}) = C_{\mathbf{g}'}(\mathfrak{p})$.

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- the infinitude of the group of units in F

Twisted First Moment

Recall: $\omega_{\mathbf{f}} = \frac{(4\pi)^{k-1} |d_F|^{1/2} \langle \mathbf{f}, \mathbf{f} \rangle_{\mathfrak{q}}}{\Gamma(k-1)}$

$$\sum_{\mathbf{f} \in \Pi_k(\mathfrak{q})} L\left(\mathbf{f} \otimes \mathbf{g}, \frac{1}{2}\right) C_{\mathbf{f}}(\mathfrak{p}) \omega_{\mathbf{f}}^{-1}$$

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$$\sum_{\mathbf{f}\in\Pi_k(\mathfrak{q})} L\left(\mathbf{f}\otimes\mathbf{g}, \frac{1}{2}\right) C_{\mathbf{f}}(\mathfrak{p})\omega_{\mathbf{f}}^{-1}$$

$$= 2\sum_{\mathfrak{m}\subset\mathcal{O}_F} \frac{C_{\mathbf{g}}(\mathfrak{m})}{\sqrt{\mathrm{N}(\mathfrak{m})}} \sum_{d=1}^{\infty} \frac{a_d(\mathfrak{n}\mathfrak{q})}{d} V\left(\frac{4^n\pi^{2n}\mathrm{N}(\mathfrak{m})d^2}{\mathrm{N}(\mathfrak{D}_F^2\mathfrak{n}\mathfrak{q})}\right) \times \sum_{\mathbf{f}\in\Pi_k(\mathfrak{q})} \omega_{\mathbf{f}}^{-1} C_{\mathbf{f}}(\mathfrak{m}) C_{\mathbf{f}}(\mathfrak{p})$$

$$V(y) = \frac{1}{2\pi i}\int_{(3/2)} y^{-u} \prod_{j=1}^{n} \frac{\Gamma\left(u+\frac{|k_j-l_j|+1}{2}\right)\Gamma\left(u+\frac{k_j+l_j-1}{2}\right)}{\Gamma\left(\frac{|k_j-l_j|+1}{2}\right)\Gamma\left(\frac{k_j+l_j-1}{2}\right)} G(u)\frac{du}{u}.$$

Petersson Trace Formula

Proposition (Torotabas, 2011)

Let $k \in 2\mathbb{Z}^n_{>0}$, and let $\mathfrak{a}$ and $\mathfrak{b}$ be fractional ideals of F. If $\alpha \in \mathfrak{a}^{-1}$ and $\beta \in \mathfrak{b}^{-1}$, we have

$$\sum_{\mathbf{f} \in \Pi_k(\mathfrak{q})} \omega_{\mathbf{f}}^{-1} C_{\mathbf{f}}(\alpha\mathfrak{a}) C_{\mathbf{f}}(\beta\mathfrak{b}) + (Oldforms) = \mathbb{1}_{\alpha\mathfrak{a}=\beta\mathfrak{b}}$$

$$+ \quad * \sum_{\substack{\bar{\mathfrak{c}}^2 = \mathfrak{a}\bar{\mathfrak{b}} \\ c \in \mathfrak{c}^{-1} \setminus \{0\} \\ \epsilon \in \mathcal{O}_F^{\times +}/\mathcal{O}_F^{\times 2}}} \frac{Kl(\epsilon\alpha, \mathfrak{a}; \beta, \mathfrak{b}; c, \mathfrak{c})}{\mathrm{N}(c\mathfrak{c})} \prod_{j=1}^{n} J_{k_j - 1}\left(\frac{4\pi\sqrt{\epsilon_j \alpha_j \beta_j [\mathfrak{a}\mathfrak{b}\mathfrak{c}^{-2}]_j}}{|c_j|} \right)$$

- $\mathfrak{m}$: integral ideal of $F \Longrightarrow \mathfrak{m} = \alpha\mathfrak{a}$ for some $\mathfrak{a} \in Cl^+(F)$ and $0 \ll \alpha \in \mathfrak{a}^{-1}$
- Similarly, $\mathfrak{p} = \beta\mathfrak{b}$ with some $\mathfrak{b} \in Cl^+(F)$ and $0 \ll \beta \in \mathfrak{b}^{-1}$.

$$\sum_{\mathbf{f}\in\Pi_k(\mathfrak{q})} L\left(\mathbf{f}\otimes\mathbf{g},\frac{1}{2}\right) C_{\mathbf{f}}(\mathfrak{p})\omega_{\mathbf{f}}^{-1} = M_{\mathfrak{p}}^{\mathbf{g}}(k,\mathfrak{q})+E_{\mathfrak{p}}^{\mathbf{g}}(k,\mathfrak{q})-E_{\mathfrak{p}}^{\mathbf{g}}(k,\mathfrak{q},\mathrm{old})$$

where

$$M_{\mathfrak{p}}^{\mathbf{g}}(k,\mathfrak{q}) = 2\frac{C_{\mathbf{g}}(\mathfrak{p})}{\sqrt{\mathrm{N}(\mathfrak{p})}}\sum_{d=1}^{\infty}\frac{a_d(\mathfrak{n}\mathfrak{q})}{d}V_{1/2}\left(\frac{4^n\pi^{2n}\mathrm{N}(\mathfrak{p})d^2}{\mathrm{N}(\mathfrak{D}_F^2\mathfrak{n}\mathfrak{q})}\right),$$

$$E_{\mathfrak{p}}^{\mathbf{g}}(k,\mathfrak{q}) = \sum_{\{\bar{\mathfrak{a}}\}}\sum_{\alpha\in(\mathfrak{a}^{-1})^{\gg 0}/\mathcal{O}_F^{\times +}}\frac{C_{\mathbf{g}}(\alpha\mathfrak{a})}{\sqrt{\mathrm{N}(\alpha\mathfrak{a})}}\sum_{d=1}^{\infty}\frac{a_d(\mathfrak{n}\mathfrak{q})}{d}V_{1/2}\left(\frac{4^n\pi^{2n}\mathrm{N}(\alpha\mathfrak{a})d^2}{\mathrm{N}(\mathfrak{D}_F^2\mathfrak{n}\mathfrak{q})}\right)$$

$$\times\sum_{\substack{\bar{\mathfrak{c}}^2=\overline{\mathfrak{a}\mathfrak{b}}\\ c\in\mathfrak{c}^{-1}\mathfrak{l}\mathfrak{q}\setminus\{0\}\\ \epsilon\in\mathcal{O}_F^{\times +}/\mathcal{O}_F^{\times 2}}}\frac{Kl(\epsilon\alpha,\mathfrak{a};\beta,\mathfrak{b};c,\mathfrak{c})}{\mathrm{N}(c\mathfrak{c})}\prod_{j=1}^{n}J_{k_j-1}\left(\frac{4\pi\sqrt{\epsilon_j\alpha_j\beta_j[\mathfrak{a}\mathfrak{b}\mathfrak{c}^{-2}]_j}}{|c_j|}\right),$$

$$E_{\mathfrak{p}}^{\mathbf{g}}(k,\mathfrak{q},\mathrm{old}) = \sum_{\mathfrak{m}\subset\mathcal{O}_F}\frac{C_{\mathbf{g}}(\mathfrak{m})}{\sqrt{\mathrm{N}(\mathfrak{m})}}\sum_{d=1}^{\infty}\frac{a_d(\mathfrak{n}\mathfrak{q})}{d}V_{1/2}\left(\frac{4^n\pi^{2n}\mathrm{N}(\mathfrak{m})d^2}{\mathrm{N}(\mathfrak{D}_F^2\mathfrak{n}\mathfrak{q})}\right)\sum_{\mathbf{f}\in\Pi_k(\mathcal{O}_F)}\frac{C_{\mathbf{f}}(\mathfrak{p})C_{\mathbf{f}}(\mathfrak{m})}{\omega_{\mathbf{f}}}$$

Level Aspect

Lemma

$$M_{\mathfrak{p}}^{\mathbf{g}}(k,\mathfrak{q}) = \frac{C_{\mathbf{g}}(\mathfrak{p})}{\sqrt{\mathrm{N}(\mathfrak{p})}}\gamma_{-1}(F)\prod_{\mathfrak{l}|\mathfrak{n}}(1-\mathrm{N}(\mathfrak{l})^{-1})\log(\mathrm{N}(\mathfrak{q})) + O(1),$$

where $\gamma_{-1}(F)$ *is the residue in the Laurent expansion of* $\zeta_F(2u+1)$ *at* $u=0$.

Lemma

We have $E_{\mathfrak{p}}^{\mathbf{g}}(k,\mathfrak{q}) = O\left(\mathrm{N}(\mathfrak{q})^{-\frac{1}{2}+\epsilon}\right)$.

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Error Term

$$E_{\mathfrak{p}}^{\mathbf{g}}(k,\mathfrak{q}) = \sum_{\{\bar{\mathfrak{a}}\}} \sum_{\alpha\in(\mathfrak{a}^{-1})^{\gg 0}/\mathcal{O}_F^{\times +}} \frac{C_{\mathbf{g}}(\alpha\mathfrak{a})}{\sqrt{\mathrm{N}(\alpha\mathfrak{a})}} \sum_{d=1}^{\infty} \frac{a_d(\mathfrak{n}\mathfrak{q})}{d} V_{1/2}\left(\frac{4^n\pi^{2n}\mathrm{N}(\alpha\mathfrak{a})d^2}{\mathrm{N}(\mathfrak{D}_F^2\mathfrak{n}\mathfrak{q})}\right)$$

$$\times \sum_{\substack{\bar{\mathfrak{c}}^2=\overline{\mathfrak{a}\mathfrak{b}} \\ c\in\mathfrak{c}^{-1}\mathfrak{q}\setminus\{0\} \\ \epsilon\in\mathcal{O}_F^{\times +}/\mathcal{O}_F^{\times 2}}} \frac{KI(\epsilon\alpha,\mathfrak{a};\beta,\mathfrak{b};c,\mathfrak{c})}{\mathrm{N}(c\mathfrak{c})} \prod_{j=1}^{n} J_{k_j-1}\left(\frac{4\pi\sqrt{\epsilon_j\alpha_j\beta_j\left[\mathfrak{a}\mathfrak{b}\mathfrak{c}^{-2}\right]_j}}{|c_j|}\right)$$

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Now consider

$$\sum_{c \in \mathfrak{c}^{-1}\mathfrak{q} \setminus \{0\}/\mathcal{O}_F^{\times +}} \sum_{\eta \in \mathcal{O}_F^{\times +}} \frac{Kl(\alpha, \mathfrak{a}; \beta, \mathfrak{b}; c\eta, \mathfrak{c})}{|\mathrm{N}(c)|} \prod_{j=1}^{n} J_{k_j - 1}\left(\frac{4\pi\sqrt{\alpha_j \beta_j [\mathfrak{a}\mathfrak{b}\mathfrak{c}^{-2}]_j}}{\eta_j |c_j|}\right)$$

J-Bessel Function

The J-Bessel function is defined as

$$J_u(x) = \int_{(\sigma)} \frac{\Gamma\left(\frac{u-s}{2}\right)}{\Gamma\left(\frac{u+s}{2}+1\right)} \left(\frac{x}{2}\right)^s \, ds \qquad x > 0,\ 0 < \sigma < \Re(u).$$

We have

$$J_u(x) \ll x^{1-\delta} \quad \text{for} \ \ 0 \le \delta \le 1.$$

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$$\prod_{j=1}^{n} J_{k_j-1}\left(\frac{4\pi\sqrt{\alpha_j\beta_j\left[\mathfrak{abc}^{-2}\right]_j}}{\eta_j|c_j|}\right) \ll \prod_{j=1}^{n}\left(\frac{\sqrt{\alpha_j\beta_j\left[\mathfrak{abc}^{-2}\right]_j}}{\eta_j|c_j|}\right)^{1-\delta_j},$$

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where $\delta_j = 0$ if $\eta_j \geq 1$, and $\delta_j = \delta$ for some fixed $\delta > 0$ otherwise.

Controlling the totally positive units

Key Observation (Luo, 2003)

$$\sum_{\eta \in \mathcal{O}_F^{\times +}} \prod_{\eta_j < 1} \eta_j^{\delta} < \infty.$$

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$$\sum_{c} \sum_{\eta} \frac{KI}{|\mathrm{N}(c)|} \prod_{j} J_{k_j - 1}$$
$$\ll \sum_{\eta \in \mathcal{O}_F^{\times +}} \prod_{\eta_j < 1} \eta_j^{\delta} \sum_{c \in \mathfrak{c}^{-1}\mathfrak{q} \setminus \{0\} / \mathcal{O}_F^{\times +}} |c|^{\delta - 1} \frac{\mathrm{N}((\alpha\mathfrak{a}, \beta\mathfrak{b}, \mathfrak{c}))^{1/2}}{\mathrm{N}(\mathfrak{c})^{3/2-\delta}}.$$

Weight Aspect

Lemma

$$M_{\mathfrak{p}}^{\mathbf{g}}(\boldsymbol{k}, \mathfrak{q}) = \frac{C_{\mathbf{g}}(\mathfrak{p})}{\sqrt{\mathrm{N}(\mathfrak{p})}} \gamma_{-1}^{\mathfrak{n}\mathfrak{q}}(F) \log \boldsymbol{k} + O(1)$$

where $\gamma^{\mathfrak{n}\mathfrak{q}}(F)$ *is the residue in the Laurent expansion of* $\zeta_F^{\mathfrak{n}\mathfrak{q}}$ *at* 1.

Lemma

$$E_{\mathfrak{p}}^{\mathbf{g}}(\boldsymbol{k}, \mathfrak{q}) = O(1).$$

Thank you!