Counting Square Discriminants

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Motivating Question

Question: How many integral binary quadratic forms $ax^2 + bxy + cy^2$ are there of fixed discriminant h and with bounded coefficients?

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- This problem lives in a larger context of counting integral points on hyperbolic surfaces.

Background

- Erdös, 1952: Mentioned an unpublished result of Bellman and Shapiro:

$$\sum_{n=1}^{X} d(f(n)) = cX \log X + O(X \log \log X)$$

where $d(k)$ counts the divisors of k and f is an irreducible quadratic polynomial.

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- Scourfield, 1961: Generalized this result and proved:

$$\sum_{n=1}^{X} r_2(f(n)) = cX \log X + O(X \log \log X)$$

where $r_2(k)$ counts the number of ways k can be written as the sum of two squares.

Background, continued

- Duke, Rudnick, Sarnak, 1993: Wanted to count integral points on the one-sheeted hyperboloid defined by

$$-x_1^2 - x_2^2 + kx_3^2 = -1$$

with $\|\mathbf{x}\| < X$.

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- Remarks:
  - D-R-S result is much more general.
  - Eskin, McMullen, 1993 have essentially the same results using ergodic methods.

Background, continued

- Oh, Shah, 2011: Let $Q(x_1, x_2, x_3)$ be an integral quadratic form with signature (2,1). Then

$$\#\{\mathbf{x} \in \mathbb{Z}^3 \mid Q(\mathbf{x}) = h,\ \|\mathbf{x}\| < X\} \sim cX \log X.$$

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- As a corollary, because the discriminant is a quadratic form with signature (2,1), we get:

Theorem (Oh, Shah)

For h a square,

$$\#\{Q(x,y) = ax^2 + bxy + cy^2 \mid \operatorname{disc} Q = h,\ a^2 + b^2 + c^2 \leq X\}$$
$$= cX \log X + O(X(\log X)^{\frac{3}{4}})$$

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Theorem (H-K-K-L)

$$\sum_{a,c=1}^{\infty} \tau(4ac+h)e^{-\left(\frac{a+c}{X}\right)} = c_1(h)X\log X + c_2(h)X + O(X^{\frac{1}{2}})$$

where $c_1(h)=0$ if h is not a square and

$$\tau(n) = \begin{cases} 0 & \text{if } n \neq \square \\ 1 & \text{if } n = 0 \\ 2 & \text{if } n = \square,\ n \neq 0 \end{cases}$$

Statement of Results, continued

Notice that

$$\sum_{a,c=1}^{\infty} \tau(4ac+h) e^{-\left(\frac{a+c}{X}\right)}$$

is a smoothed sum of

$$\sum_{a,c=1}^{X} \tau(4ac+h)$$

$$= \#\{(a,b,c) \in \mathbb{Z}^3 \mid b^2 - 4ac = h,\ 1 \leq a, c \leq X,\ |b/2| \leq X\}.$$

Statement of Results, continued

We also have

Theorem (H-K-K-L)

$$\sum_{a,c=1}^{X} \tau(4ac+h) = c_1(h)X\log X + c_2(h)X + O(X^{\frac{34}{39}}).$$

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If we use this condition and count integral points in a hyperbolic region instead of a box, we get an error term of $O(X^{\frac{4}{5}})$.

Proof Sketch

- We study the Dirichlet series

$$\sum_{a,c} \frac{\tau(4ac+h)}{a^w c^s}.$$

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- Note:

$$\sum_{a,c} \frac{\tau(4ac+h)}{a^w c^s} = \sum_{m=1}^{\infty} \frac{\tau(4m+h)}{m^s} \sum_{a|m} \frac{1}{a^v}$$

$$= \sum_{m=1}^{\infty} \frac{\tau(4m+h)\sigma_{-v}(m)}{m^s}$$

Proof Sketch, continued

Conclusion: We want to study the *Shifted Convolution Sum*:

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Key idea: Use Fourier coefficients of automorphic forms.

Proof Sketch, continued

Conclusion: We want to study the *Shifted Convolution Sum*:

$$\sum_{m=1}^{\infty} \frac{\tau(4m+h)\sigma_{-v}(m)}{m^s}.$$

Key idea: Use Fourier coefficients of automorphic forms.

- $\theta(z)$ gives the square indicator function, $\tau(n)$.
- The Eisenstein series $E(z, \frac{1+v}{2})$ gives the divisor function, $\sigma_{-v}(m)$.

Proof: Key Points

- Use a half-integral weight version of the Poincare series studied by Hoffstein and Hulse. For f, g weight k modular forms,

$$\langle P_h, \bar{f} g y^k \rangle = (\Gamma \text{ factors}) \sum_{m=1}^{\infty} \frac{a(m+h)b(m)}{m^{s+k-1}}.$$

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We use $P_h(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} \Im(\gamma z)^s e^{-2\pi i h z} \frac{j(\gamma,z)}{|j(\gamma,z)|}$.

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- If we unfold the integral of the inner product, we get the Dirichlet series we want.
- Use the spectral expansion of the Poincare series to locate poles and compute residues.
- Take inverse Mellin transforms and shift lines of integration.

Proof: Difficulties

There are two main difficulties.

- $V(z) = \theta(z)E(4z, \frac{1+v}{2})y^{1/4}$ is not L^2 (it has moderate growth). To correct this, we actually take the inner product of P_h with $V(z)-$(lin. combo. of 1/2-integral wt. Eisenstein series).

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- P_h is not L^2 (it has exponential growth in y). To get around this, cut off P_h and take limits carefully at the end.

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