New upper bounds for the number of divisors function

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(joint work with Jean-Marie De Koninck)

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Introduction

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- If the factorization in distinct prime factors is

$$n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \quad (p_1 < \cdots < p_k)$$

then we say that

$$(\alpha_1, \ldots, \alpha_k)$$

is the *exponent vector* of n.

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- For each $\epsilon > 0$, there is a constant $C(\epsilon)$ such that
$$\tau(n) \leq C(\epsilon) n^{\epsilon}.$$
In fact, we have
$$C(\epsilon) := \prod_{p < 2^{1/\epsilon}} \max_{\alpha \geq 0} \frac{\alpha + 1}{p^{\alpha\epsilon}}.$$

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- Nicolas and Robin (1983) have shown that

$$\tau(n) \le 2^{\eta_1\frac{\log n}{\log\log n}} \quad \text{for each } n \ge 3$$

where $\eta_1 := 1.53793986\ldots$ with equality only for $n = 6983776800$.

Using the arithmetic geometric mean inequality

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- We have

$$
\begin{aligned}
(\alpha_1+1)\cdots(\alpha_k+1) &\le \left(\frac{\alpha_1+1+\cdots+\alpha_k+1}{k}\right)^k \\
&= \left(\frac{k+\alpha_1+\ldots+\alpha_k}{k}\right)^k.
\end{aligned}
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$$(\alpha_1+1)\cdots(\alpha_k+1) \le \left(\frac{\alpha_1+1+\cdots+\alpha_k+1}{k}\right)^k = \left(\frac{k+\alpha_1+\ldots+\alpha_k}{k}\right)^k.$$

- Thus

$$\tau(n) \le \left(1+\frac{\Omega(n)}{\omega(n)}\right)^{\omega(n)} \quad (n \ge 2).$$

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- We deduce that for each $\epsilon > 0$ we have

$$2^{\omega(n)} \le \tau(n) \le (2+\epsilon)^{\omega(n)}$$

for almost all $n \le x$ as $x \to \infty$.

Best possible inequalities 1

Theorem

For every integer $n \geq 2$,

$$\tau(n) \leq \left(\frac{\eta_2 \log n}{\omega(n) \log_+ \omega(n)} \right)^{\omega(n)},$$

where

$$\eta_2 := \exp\left(\frac{1}{6} \log 96 - \log\left(\frac{\log 60060}{6 \log 6} \right) \right) = 2.0907132\ldots$$

and $\log_+(z) := \log(\max(2, z))$.

Best possible inequalities 2

Theorem

For every integer $n \geq 2$,

$$\tau(n) \leq \left(1 + \eta_3 \frac{\log n}{\omega(n) \log_+ \omega(n)}\right)^{\omega(n)}$$

where

$$\eta_3 := \frac{(1152^{1/7} - 1)7 \log 7}{\log 367567200} = 1.1999953\ldots$$

An inequality for large integers

Theorem

For every integer $n > 7821398034525610735 20$,

$$\tau(n) < \left(\frac{2\log n}{\omega(n)\log_+ \omega(n)}\right)^{\omega(n)}.$$

Moreover, the inequality remains true for all $n \geq 2$ *with* $\omega(n) \leq 3$.

The main result

Theorem

For every positive integer n with $\omega(n) \geq 74$,

$$\tau(n) < \left(1 + \frac{\log n}{\omega(n) \log \omega(n)}\right)^{\omega(n)}.$$

Comments on the result 1

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- We introduce the function $\lambda(n)$ defined implicitly by

$$\tau(n) = \left(1 + \frac{\lambda(n)\log n}{\omega(n)\log\omega(n)}\right)^{\omega(n)},$$

when $\omega(n) \geq 2$. Therefore, for each integer $n \geq 2$ with $\omega(n) \geq 2$, we set

$$\lambda(n) := \frac{(\tau(n)^{1/\omega(n)} - 1)\omega(n)\log\omega(n)}{\log n}.$$

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$$\lambda(n) := \frac{(\tau(n)^{1/\omega(n)} - 1)\omega(n)\log\omega(n)}{\log n}.$$

- $$\tau(n) < \left(1 + \frac{\log n}{\omega(n)\log\omega(n)}\right)^{\omega(n)} \iff \lambda(n) < 1$$

Comments on the result 2

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- The Theorem is best possible since the integer

$$n_0 = 2^{13} \cdot 3^8 \cdot 5^5 \cdot 7^4 \cdot 11^3 \cdot 13^3 \cdot 17^3 \cdot 19^2 \cdots 53^2 \cdot 59 \cdots 367$$

satisfies $\omega(n_0) = 73$ and $\lambda(n_0) = 1.0008832\ldots$ They are many other examples but n_0 is the unique such integer that maximises the function λ.

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  satisfies $\omega(n_0) = 73$ and $\lambda(n_0) = 1.0008832\ldots$ They are many other examples but n_0 is the unique such integer that maximises the function λ.
- Using the same methods, we can show that

  $$n_1 = 2^{13} \cdot 3^8 \cdot 5^5 \cdot 7^4 \cdot 11^3 \cdot 13^3 \cdot 17^3 \cdot 19^2 \cdots 53^2 \cdot 59 \cdots 373$$

  is the only integer that realises the maximum of λ when restricted to $\omega(n) = 74$. We have $\lambda(n_1) = 0.99991077\ldots$

Comments on the result 3

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- There are only finitely many integers n with $\omega(n) \geq 44$ such that $\lambda(n) > 1$. In fact, none of these numbers n exceeds $\exp(10758.21)$. One such large n with $\omega(n) = 44$ is the one whose exponent vector is

$$(354, 223, 152, 125, 102, 95, 86, 83, 77, 72, 71, 67, 65, \\ 64, 63, 61, 59, 59, 57, 57, 56, 55, 55, 54, 53, 52, 52, 52, \\ 51, 51, 50, 49, 49, 49, 48, 48, 48, 47, 47, 47, 46, 46, 46, 46),$$

and this number n has 4622 digits and its size is about $\exp(10640.84)$. Moreover, it can be established that any other integer n with $\lambda(n) > 1$ and $\omega(n) \geq 45$ is less than $\exp(4569.68)$.

Fondamental inequality

Lemma (Somasundaram (1987))

For every integer $n \geq 2$,

$$\tau(n) \leq \left(\frac{\log(n\gamma(n))}{\omega(n)}\right)^{\omega(n)} \prod_{p|n} \frac{1}{\log p}.$$

Key lemma

Lemma

Let $n \geq 2$ *be an integer and* p *a prime number. If* $p^\alpha \| n$ *with* $\alpha \geq 2$*, then*

$$\frac{\lambda(n)}{\lambda(n/p)} \leq \left(1 + \frac{2}{\alpha\omega(n)}\right)\left(1 - \frac{\log p}{\log n}\right).$$

Elementary estimates

Lemma

We have

$$\sum_{i=1}^{k} \log p_i \leq k(\log k + \log\log k - 1/2) \qquad \textit{for} \quad k \geq 5,$$

$$\sum_{i=1}^{k} \log\log p_i \geq k\left(\log\log k + \frac{\log\log k - 3/2}{\log k}\right) \qquad \textit{for } k \geq 6$$

and

$$\prod_{i=1}^{k} \frac{1}{\log p_i} < (\log k)^{-k} \qquad \textit{for } k \geq 44.$$

Sketching the proof of the main theorem 1

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- For obvious reasons, the largest values of λ are acheived by integers of the form

$$\prod_{i=1}^{k} p_i^{\alpha_i} \qquad \text{with } \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k,$$

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  where the p_i's are the primes in ascending order.
- Using the fundamental inequality and the elementary estimates, we rule out the cases with $k > 94$.
- Also, again with the fundamental inequality, we get an upper bound for any possible counterexample n with $74 \leq \omega(n) \leq 94$.

Sketching the proof of the main theorem 2

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- We specialise the structure of the possible counterexample n to

$$n := p_1^{\alpha_1} \cdots p_{j_2}^{\alpha_{j_2}} \cdot p_{j_2+1}^2 \cdots p_{j_1}^2 \cdot p_{j_1+1} \cdots p_k.$$

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- The upper bound and the structure limit the possible values for (j_1, j_2).
- Using the multiplicativity of τ and the fundamental inequality we are lead to define the function

$$f_1(j_2, j_1, k, z) := \frac{(c_1(j_2, j_1, k)(\log z + c_2(j_2, j_1, k))^{j_2/k} - 1)k \log k}{\log z}.$$

Sketching the proof of the main theorem 3

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- This function has the property

$$\lambda(n) \leq f_1(j_2, j_1, k, n) \leq \max_{z > -c_2(j_2, j_1, k)} f_1(j_2, j_1, k, z).$$

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- We do all the computations and keep the couple (j_1, k) only if we obtain $f_1 \geq 1$ for some $j_2 \leq j_1$.
- With the remaining possibilities, we do the same computation with one more variable on the integers

$$n = p_1^{\alpha_1} \cdots p_{j_3}^{\alpha_{j_3}} \cdot p_{j_3+1}^3 \cdots p_{j_2}^3 \cdot p_{j_2+1}^2 \cdots p_{j_1}^2 \cdot p_{j_1+1} \cdots p_k.$$

Sketching the proof of the main theorem 4

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- We have to do this step four times. For the third and the fourth step, we use the key lemma to bound the largest prime that can divide, up to the power 4, 5 and 6, the possible counterexample n that realises the maximum of λ.

Sketching the proof of the main theorem 4

- We have to do this step four times. For the third and the fourth step, we use the key lemma to bound the largest prime that can divide, up to the power 4, 5 and 6, the possible counterexample n that realises the maximum of λ.
- The other results use the same type of ideas.

Further remarks 1

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- One can show that

$$\sum_{n\le x}\left|\lambda(n)-\frac{\log\log x\log\log\log x}{\log x}\right|^2 \ll \frac{x\log\log x(\log\log\log x)^2}{\log^2 x},$$

from which we conclude that for almost all $n \le x$,

$$\lambda(n)=(1+o(1))\frac{\log\log x\log\log\log x}{\log x}\quad (x\to\infty).$$

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  from which we conclude that for almost all $n\le x$,

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- There are finitely many numbers n with $\omega(n)>43$ for which $\lambda(n)\ge 1$.
- The set of limit points of $\lambda(n)$ is the interval $\left[0,\left(\prod_{i=1}^{6}\frac{1}{\log p_i}\right)^{1/6}\log 6\right]=[0,1.145206\ldots]$.

Further remarks 2

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- We have

$$|\{n \le x : \lambda(n) \ge 1\}| = (\eta_4 + o(1)) \log^{43} x$$

where η_4 is some absolute constant.

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- By using the special numbers $\prod_{i=1}^{k} p_i$ for the lower bound and the fundamental inequality for the upper bound, we obtain

$$\sup_{\omega(n)=k} \lambda(n) = 1 - \frac{\log\log k - 1}{\log k} + \frac{(\log\log k)^2 - 3\log\log k}{\log^2 k} + O\left(\frac{1}{\log^2 k}\right) \quad (k \to \infty).$$

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