Averaging Average Orders

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Classical Problems

For $m \in \mathbb{Z}_{\geq 0}$, let

$$r(m) = \#\{m = x^2 + y^2 \mid x, y \in \mathbb{Z}\}$$

be the sum of squares function, which counts the number of integer points on the circle of radius $\sqrt{m}$ centered at the origin.

Gauss's Circle Problem is concerned with estimating $P(x)$, for $x > 1$, where $P(x)$ is defined as an error term of the sum,

$$\sum_{m \leq x} r(m) = \pi x + P(x),$$

which counts the number of lattice points inside the circle of radius $\sqrt{x}$.

Specifically, it is concerned with the smallest θ such that $P(x) = O(x^{\theta+\varepsilon})$ for all $\varepsilon > 0$. Gauss was able to prove that $\theta \leq \frac{1}{2}$.

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Similarly, let

$$d(m) = \sum_{d|m} 1$$

be the classical divisor function which counts the number of lattice points (a, b) on the hyperbola $ab = m$.

Dirichlet's Divisor Problem is concerned with estimating $\Delta(x)$, for $x > 1$, where $\Delta(x)$ is defined as an error term of the sum,

$$\sum_{m \leq x} d(m) = x \log x - x(2\gamma - 1) + \Delta(x).$$

which counts the number of lattice points (a, b) under the hyperbolic region given by $ab \leq x$.

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In 1915-1916, Hardy and Landau proved independently that for both $P(x)$ and $\Delta(x)$, $\theta \geq \frac{1}{4}$ and conjectured that indeed $P(x), \Delta(x) = O(x^{\frac{1}{4}+\varepsilon})$ for all $\varepsilon > 0$.[5]

Currently the best-known progress in this direction in both cases is due to Huxley, with $\theta = 131/416 \approx 0.3149$, with which was proven in 2000[6] for $P(x)$ and in 2003[7] for $\Delta(x)$.

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In 1916[4], Hardy was able to show this conjecture is true *on average* by proving that

$$\frac{1}{x}\int_1^x |P(t)|\,dt = O(x^{\frac{1}{4}+\varepsilon}) \qquad \text{and} \qquad \frac{1}{x}\int_1^x |\Delta(t)|\,dt = O(x^{\frac{1}{4}+\varepsilon}).$$

Cramér was able to remove the ε term in the above result in 1922[2] by obtaining an asymptotic of the mean square, showing that

$$\frac{1}{x}\int_1^x |P(t)|^2\,dt = c_1 x^{\frac{1}{2}} + O(x^{\frac{1}{4}+\varepsilon}),$$

and

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More generally, in analytic number theory we are often concerned with finding the **Average Order** of arithmetic functions.

That is, for an arithmetic function $\lambda : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ we want to better understand, for $x > 1$, the asymptotic behavior of

$$S(x) := \sum_{n \leq x} \lambda(n).$$

When these $\lambda(n)$ arise from Dirichlet series with functional equations, $S(x)$ tends to have a well-understood main term, due to the poles of the Dirichlet series, and an error term, $E(x)$, which is analogous to $P(x)$ and $\Delta(x)$.

In this case, an extremely general 1964 paper[1] due to Chandrasekharan and Narasimhan provides an asymptotic result for the mean square of $E(x)$ that is determined mainly by the analytic conductor of the Dirichlet series.

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Automorphic Forms

Let $f, g : \mathbb{H} \to \mathbb{H}$ be holomorphic cusp forms of integer or half-integer weight k and level $N \in \mathbb{N}$ with Fourier expansions

$$f(z) = \sum_{n=1}^{\infty} a(n)e(nz), \quad g(z) = \sum_{n=1}^{\infty} b(n)e(nz).$$

From these we can define the L-functions of f and g by the Dirichlet series

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^{s+(k-1)/2}}, \quad L(s, g) = \sum_{n=1}^{\infty} \frac{b(n)}{n^{s+(k-1)/2}},$$

when $\Re(s) > 1$, which have analytic continuations to all $s \in \mathbb{C}$ and satisfy functional equations of the type $s \to 1 - s$.

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We would like to investigate the average order of the Fourier coefficients of f. We define

$$S_f(x) := \sum_{m \leq x} a(m).$$

Since $L(s, f)$ is entire, we do not have a main term for $S_f(x)$ and so we can take $S_f(x)$ to be its own error term. As a corollary of Chandrasekharan and Narasimhan's result, we have that,

$$\int_1^x |S_f(y)|^2 \, dy = cx^{k+\frac{1}{2}} + O(x^k \log^2 x).$$

This was proven earlier by Walfisz[9] in the case of the Ramanujan τ-function.

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Using this with the Cauchy-Schwarz inequality tells us that

$$\frac{1}{x}\int_1^x |S_f(y)|\,dy = O(x^{\frac{k-1}{2}+\frac{1}{4}})$$

which is an *on average* result of what Hafner and Ivić refer to as the **classical conjecture.**[3]

Conjecture

For all $\varepsilon > 0$,

$$S_f(x) = O(x^{\frac{k-1}{2}+\frac{1}{4}+\varepsilon}).$$

If we used normalized coefficients, $A(n) = a(n)n^{\frac{1-k}{2}}$, instead then this conjecture states that the average order grows like $O(x^{\frac{1}{4}+\varepsilon})$, which is completely analogous to the expected conjectures for Gauss's Circle Problem and the Dirichlet Divisor Problem.

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This conjecture suggests "amazing cancellation"[David] of the Fourier coefficients, far more than one expects for a random variable.

In 1989[3], Hafner and Ivić were able to show, when $f(z)$ is an integral weight holomorphic cusp form,

$$S_f(x) = O(x^{\frac{k-1}{2}+\frac{1}{3}+\varepsilon}).$$

They also stated the following conjecture, and claimed that the classical conjecture followed as a corollary of the following conjecture about the mean-square.

Conjecture (Hafner and Ivić)

For all $\varepsilon > 0$,

$$\int_1^x |S_f(y)|^2 \, dy = cx^{k+\frac{1}{2}} + O(x^{k-\frac{1}{4}+\varepsilon}).$$

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Average Order Dirichlet Series

We became interested in this story when we first considered sums of the form

$$\sum_{n\le x} |S_f(n)|^2 = \int_1^x |S_f(y)|^2 \, dy + O(x^{k-\frac{1}{3}})$$

This arose when we were pondering a question about sign changes of $S_f(n)$ posed by Jeffrey Hoffstein to Winfried Kohnen at the former's birthday conference.

Inspired by the Rankin-Selberg L-function for automorphic forms, we considered a Dirichlet series of the form

$$\sum_{n=1}^{\infty} \frac{|S_f(n)|^2}{n^{s+k-1}}.$$

with the hope that we could make progress towards the mean-square conjecture by taking an inverse Mellin transform of this series.

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Average Order Dirichlet Series

Indeed, by decomposing the above series and series like it into shifted convolution sums, we were able to prove the following.

Theorem (H., Kuan, Lowry-Duda, Walker)

Let $f, g : \mathbb{H} \to \mathbb{H}$ be holomorphic cusp forms of integer or half-integer weight k and level $N \in \mathbb{N}$. Let $S_f(x)$, $S_g(x)$ as defined above. We have that the series,

$$D(s, S_f \times \overline{S_g}) := \sum_{n=1}^{\infty} \frac{S_f(n)\overline{S_g(n)}}{n^{s+k-1}} \quad \text{and} \quad D(s, S_f \times S_g) := \sum_{n=1}^{\infty} \frac{S_f(n)S_g(n)}{n^{s+k-1}},$$

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Let $f, g : \mathbb{H} \to \mathbb{H}$ be holomorphic cusp forms of integer or half-integer weight k and level $N \in \mathbb{N}$. Let $S_f(x)$, $S_g(x)$ as defined above. We have that the series,

$$D(s, S_f \times \overline{S_g}) := \sum_{n=1}^{\infty} \frac{S_f(n)\overline{S_g(n)}}{n^{s+k-1}} \quad \textit{and} \quad D(s, S_f \times S_g) := \sum_{n=1}^{\infty} \frac{S_f(n)S_g(n)}{n^{s+k-1}},$$

which are absolutely convergent for $\Re(s) > \frac{3}{2}$, each have a meromorphic continuation to all $s \in \mathbb{C}$.

Average Order Dirichlet Series

By taking inverse Mellin transforms, we are indeed able to obtain asymptotic results for sharp sums of the form

$$\sum_{n\leq x} S_f(n)\overline{S_g(n)} \quad \text{and} \quad \sum_{n\leq x} S_f(n)S_g(n).$$

However, in the cases where results like these are known we are presently unable to improve upon Chandrasekharan and Narasimhan's result. However, our Dirichlet series construction is able to tell us whether the terms in the asymptotic expansion arise from singularities. We are able to characterize this information in the following theorem which gives a smooth analog of the above sums.

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Theorem (H., Kuan, Lowry-Duda, Walker)

For any $\epsilon > 0$, and f and g are integral weight modular forms, as above,

$$\sum_{n=1}^{\infty} S_f(n)\overline{S_g(n)}e^{-n/x} = \sum_j C_j x^{k+\frac{1}{2}+it_j} + Cx^{k+\frac{1}{2}} + O_{f,g,\epsilon}(x^{k-\frac{1}{2}+\theta+\epsilon})$$

and

$$\sum_{n=1}^{\infty} S_f(n)S_g(n)e^{-n/x} = \sum_j D_j x^{k+\frac{1}{2}+it_j} + Dx^{k+\frac{1}{2}} + O_{f,g,\epsilon}(x^{k-\frac{1}{2}+\theta+\epsilon})$$

where the C_j, D_j and t_j come from a basis of Maass forms, with all $C_j = D_j = 0$ if $f(z) = cg(z)$ for some $c \in \mathbb{C}$. The constants C and D can be computed from special values of a relevant Rankin-Selberg L-function.

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We expect this result to hold for the half-integral weight case though we have not actually worked out all of the details at present. In particular, we currently have no obvious reason for the spectral terms to vanish in the case of $|S_f(n)|^2$ like they do for integral-weight forms, except that it would be in agreement with Chandrasekharan and Narasimhan's result.

Along similar lines, we really expect these spectral terms to vanish in all of the above expansions and we are surprised that this can only be deduced so far in the well-studied cases.

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We are able to get the continuation of our Dirichlet series by considering the following decomposition.

$$D(s, S_f \times \overline{S_g}) = \sum_{n\geq 1} \frac{S_f(n)\overline{S_g(n)}}{n^{s+k-1}} = \sum_{n=1}^{\infty} \frac{1}{n^{s+k-1}} \sum_{m=1}^{n} a(m) \sum_{h=1}^{n} \overline{b(h)}.$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{s+k-1}} \left(a(n)\overline{b(n)} + \sum_{\ell\geq 1} a(n)\overline{b(n-\ell)} + \sum_{\ell\geq 1} \overline{b(n)}a(n-\ell) \right)$$

$$+ \sum_{n,m\geq 1} \frac{1}{(n+m)^{s+k-1}} \left(a(n)\overline{b(n)} + \sum_{\ell\geq 1} a(n)\overline{b(n-\ell)} + \sum_{\ell\geq 1} \overline{b(n)}a(n-\ell) \right).$$

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From here, we are able to use what we know about continuing shifted convolution sums via spectral expansion. What is surprising is that, in the integral weight case, all of the problematic poles vanish. Those due to the Rankin-Selberg L-functions cancel out in the above expansions and the spectral poles vanish due to trivial zeros of L-functions for even Maass forms or the factorization of triple products involving odd Maass forms.

That is, for u_j a Maass form of weight zero of level N, and eigenvalue $\frac{1}{4} + t_j^2$ we have that

$$L(\pm it_j, \mu_j) = 0$$

when u_j is an even Maass form and

$$\langle y^k |f|^2, u_j \rangle = \langle y_k f T_{-1} f, u_j \rangle = 0$$

when u_j is odd. It is not at all obvious that $\langle y^k |f|^2, u_j \rangle$ vanishes when f is a half-integral weight form, or that $\langle y^k f \overline{g}, u_j \rangle = 0$ or $\langle y^k f T_{-1} g, u_j \rangle = 0$ when $f(z) \neq cg(z)$.

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Average Order Dirichlet Series

We are still studying these objects. We are able to use these sums to get estimates of average orders in small envelopes around x

$$\sum_{|n-x|<y} |S_f(n)|^2$$

We are also to get information about sign-changes of $S_f(n)$ using the Meher-Murty axiomatization[8] when these $a(n)$ are real. Specifically we can show that for n in the interval $[x, x+x^{\frac{3}{4}}]$, $S_f(n)$ changes sign at least once. We are trying to generalize this to the case where $a(n)$ are not real.

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Thanks!

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