On Some Constructions using Marked Ruler and Compass

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Joint work with
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**Happy Birthday,
Hershy and Manfred!!**

This is a report on some recent work:

- *On the Construction of the Regular Hendecagon by Marked Ruler and Compass*, Math. Proc. Camb. Phil. Soc., vol. **156** (2014), 409-424.

and

- *Some Fifth Roots that are Constructible by Marked Ruler and Compass*, Rocky Mountain J. of Math. (to appear)

Marked Ruler and Compass Constructions.

Tools of the trade:

Compass: Given 2 pts., may draw a circle centered at one point passing through the other.

Marked ruler (with 2 marks 1 unit apart):

(i) *as an unmarked ruler:* May draw line through 2 given pts.

(ii) *as a verging tool:* Given pt. V and curves, C_1, C_2, determine pts. P_i on C_i such that the P_i are one unit apart and the line through the P_i passes through V. The P_i are said to be *constructed by verging through V between the curves C_i.*

MC-Constructible point:

is a point in $\mathbb{R}^2$ that is obtained from the "initial set" $\{(0,0),(1,0)\}$ by repeated use of the marked ruler and compass.

MC-Constructible number:

is the x-coordinate of an MC-constructible point.

General Problem:

Give a "useful" characterization of the field of MC-constructible numbers.

Some Simpler Cases:

I. The Classical Case.

Using compass and unmarked ruler:

Characterization Theorem.

Let $x \in \mathbb{R}$. Then

(a) x is constructible by compass and unmarked ruler
$\Leftrightarrow$

(b) $\exists$ fields K_i s.t. $\mathbb{Q} = K_0 \subset \cdots \subset K_n \subset \mathbb{R}$, $x \in K_n$ and $[K_{i+1} : K_i] = 2$.

Application: Regular n-gons are constructible if and only if $n = 2^a p_1 \cdots p_k$ where p_i are distinct Fermat primes, ($p = 2^m + 1$).

II. Verging between Lines.

Using compass and marked ruler, (as an unmarked ruler, but) verging only between lines:

Characterization Theorem.

Let $x \in \mathbb{R}$; then

(a) x is constructible by compass and marked ruler, verging only between lines
$\Leftrightarrow$

(b) $\exists$ fields K_i s.t. $\mathbb{Q} = K_0 \subset \cdots \subset K_n \subset \mathbb{R}$, $x \in K_n$ and $[K_{i+1} : K_i] = 2, 3$.

Application: Regular n-gons are constructible if and only if $n = 2^a 3^b p_1 \cdots p_k$ where p_i are distinct Pierpont primes, ($p = 2^\ell 3^m + 1$).

III. RMC-Constructions.

Using **C**ompass and **M**arked ruler with verging **R**estricted between pairs of lines or a line and a circle
(no verging between circles).

Consider verging between a line and a circle:

Want the verging points to define an algebraic set:

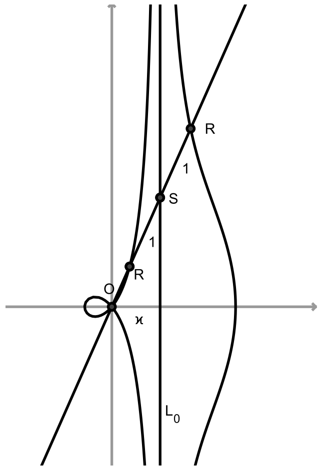
The conchoid of Nicomedes:

Given verging pt. $V = (0, 0)$ and a fixed vertical line $L_0 : x = a > 0$,

$Con_{L_0,V} : r = a \sec\theta \pm 1$,

is the conchoid through V with axis L_0.

.



$$Con_{L_0,V} : (x-a)^2(x^2+y^2) = x^2.$$

The verging points:

$Con : (x-a)^2(x^2+y^2) = x^2$
$\cap$
$Cir : (x-b)^2 + (y-c)^2 = s^2$
$\Rightarrow$
x-coordinate of points of $\cap$ satisfies a degree 6 poly.

Change of variable: $z = \frac{x}{x-a}$.

If $(x,y) \in Con$, then $z^2 = x^2 + y^2$.

z is the "signed distance" between V and (x, y).

Verging Theorem (Part I).

If (x, y) is in $Con \cap Cir$, then z is a root of the "verging poly (with parameters a, b, c, s)"

$$f(X) = X^6 + a_1 X^5 + \cdots + a_5 X + a_6,$$

where

$$\begin{aligned}
a_1 &= -2, \\
a_2 &= 1 - 2(s^2 - b^2 + c^2 + 2ab), \\
a_3 &= 4(s^2 - b^2 + c^2 + ab), \\
a_4 &= (s^2 - b^2 + c^2 + 2ab)^2 - 2(s^2 - b^2 + c^2) - \\
&\quad\ 4c^2(s^2 - (a - b)^2), \\
a_5 &= 2(ab)^2 - 2(s^2 - b^2 - c^2 + ab)^2, \\
a_6 &= (s^2 - b^2 - c^2)^2.
\end{aligned}$$

Verging Theorem (Part II).

$f(X) = X^6 + a_1 X^5 + \cdots + a_6 \in \mathbb{R}[X]$ is a verging poly with parameters $a_\varepsilon, b_\varepsilon, c_\varepsilon, s_\varepsilon$, if

- $a_1 = -2\,;$
- $(2a_6 + a_5)^2 = m^2 a_6 \quad (m = 2 - 2a_2 - a_3)\,;$
- $a_3 > -\varepsilon\sqrt{m^2 - 8a_5}\,, \quad (\varepsilon = \pm 1) \quad \Rightarrow$

. $\quad c = c_\varepsilon = \sqrt{\frac{1}{8}(a_3 + \varepsilon\sqrt{m^2 - 8a_5})}\,;$

- $m > -\frac{B}{2c_\varepsilon^2} \quad (B = \ldots) \Rightarrow$

. $\quad a = a_\varepsilon = \sqrt{\frac{m}{4} + \frac{B}{8c_\varepsilon^2}}\,, \;\; b = b_\varepsilon = \frac{m}{4a_\varepsilon}\,;$

- $\frac{m^2}{a_\varepsilon^2} > 16c_\varepsilon^2 + 8(1 - a_2 - a_3) \quad \Rightarrow$

. $\quad s = s_\varepsilon = \sqrt{\frac{m^2}{16a_\varepsilon^2} - \frac{1}{2}(1 - a_2 - a_3) - c_\varepsilon^2}\,;$

- $(s_\varepsilon^2 - b_\varepsilon^2 - c_\varepsilon^2)^2 = a_6\,.$

A Characterization Theorem.

Let $x \in \mathbb{R}$. Then

x is an RMC number
$\Leftrightarrow$

$\exists$ fields K_i s.t. $\mathbb{Q} = K_0 \subset \cdots \subset K_n \subset \mathbb{R}$, $x \in K_n$ and $[K_{i+1} : K_i] \leq 6$
and
$[K_{i+1} : K_i] = 5, 6 \Rightarrow K_{i+1} = K_i(z_{i+1})$ for some signed distance z_{i+1} corresponding to a point of $\cap$ of a Con and Cir with parameters in K_i.

We'll call $\{K_j\}_j$ an RMC tower.

Notice $\sqrt[7]{2}$ is not RMC (actually not MC) nor is the regular 23-gon.

Is $\sqrt[5]{2}$ an RMC number? Not known.

Our Main Application.

$2\cos\frac{2\pi}{11}$ is an RMC number.

Hence a regular 11-gon is constructible by marked ruler and compass.

"Idea" of the proof:

We found that $\alpha := 2\cos\frac{2\pi}{11}$ is contained in an RMC tower of length 2:

$$\mathbb{Q} \subset K_1 \subset K_2\,,$$

with

$[K_1 : \mathbb{Q}] = 3$ and $[K_2 : K_1] = 5$.

Since $K_2 = K_1(\alpha)$ and of degree 5 over K_1 we wanted to find a generator z_2

$$K_1(\alpha) = K_2 = K_1(z_2)\,,$$

s.t. z_2 is the root of a verging poly over K_1.

In general

$$z_2 = u\alpha + u_2\alpha_2 + u_3\alpha_3 + u_4\alpha_4 + u_5\alpha_5$$

where $\alpha, \ldots, \alpha_5$ form a basis of K_2/K_1.

Simplifying assumption: Take $z_2 = u\alpha$.

The Verging Theorem implies u is a real root of

$g(x) :=$

$$5x^{12}+22x^{11}+48x^{10}+76x^{9}+84x^{8}+64x^{7}+36x^{6}+8x^{5}$$

Miracle!!!

$$g(x) = x^5(5x+2)(x^3+2x^2+2x+2)^2,$$

and taking u to be the real root of the irreducible cubic works.

By taking $K_1 = \mathbb{Q}(u)$, we get the RMC tower:

$$\mathbb{Q} \subset \mathbb{Q}(u) \subset \mathbb{Q}(u, z_2) = \mathbb{Q}(u, \alpha)\,.$$

Selected References.

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