Infinitely Many New Quadratic Number Fields with 2-Class Group of Rank 4 Having Infinite 2-Class Field Tower

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August, 2014
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- Given an algebraic number field k, we denote by k^1 the Hilbert 2-class field of k, i.e. the maximal abelian unramified extension of k with degree a power of 2. For non-negative integers n we define the Hilbert 2-class field k^n inductively as $k^0 = k$ and $k^{n+1} = (k^n)^1$. Denoting by C the containment symbol, we define $k^0 \subset k^1 \subset k^2 \subset \ldots k^n \subset \ldots$ to be the 2-class field tower of k. We say that the tower is finite if $k^n = k^{n+1}$ for some n, with length n if n is minimal, and infinite otherwise.

- It is well known that if k is an imaginary quadratic number field with the rank of 2-class group $C_{k,2}$, i.e. the 2-Sylow subgroup of the ideal class group C_k (in the wide sense) of k, greater than or equal to 5, then k has infinite 2-class field tower.

- It is also well known that for k imaginary quadratic with rank $C_{k,2} = 2$ or 3, then the 2-class field tower of k may be finite or infinite, and that if rank $C_{k,2} = 1$ then the 2-class field tower of k is finite and has length 1.

- It was conjectured in the late 1970s that if k is imaginary quadratic with rank $C_{k,2} = 4$, then k has infinite 2-class field tower. From our earlier work (Benjamin, 2001, 2002), in addition to the work of Hajir (1996, 2000), Mouhib (2010), and Sueyoshi (2004, 2009, 2010), we know that if k is as above then k has infinite 2-class field tower in the following cases:

- A) 4-rank of C_k greater than or equal to 3 (Hajir, 1996, 2000)

- B) exactly one negative prime discriminant divides the discriminant d_k of k (Mouhib, 2010)

- C) 4-rank of C_k equal to 2 and five negative prime discriminants divide d_k (Benjamin, 2002)

- D) 4-rank of C_k equal to 2, exactly three negative prime discriminants divide d_k, and d_k not congruent to 4 mod 8 (Benjamin, 2002)

- E) 4-rank of C_k equal to 1, five negative prime discriminants divide d_k, and d_k not congruent to 4 mod 8 (Sueyoshi, 2009)

- **NOTE**: For the remainder of this talk, unless stated otherwise k will always denote an imaginary quadratic number field with rank $C_{k,2} = 4$.

- In the case when the 4-rank of C_k is equal to 2, exactly three negative prime discriminants divide d_k, and d_k is congruent to 4 mod 8, we have shown (Benjamin, 2002) that k has infinite 2-class field tower except for one particular family with certain specified Kronecker symbols of the primes dividing d_k, which has been corroborated by Sueyoshi (2004, 2010) using Rédei matrices.

- The above specified Kronecker symbols are as follows, where p_1 and p_2 are distinct primes dividing positive prime discriminants dividing d_k, and $q_1 = 2$ (wlog), q_2, and q_3 are distinct primes dividing negative prime discriminants dividing d_k: $(p_1/q_1) = (p_1/q_2) = (p_1/q_3) = (p_2/q_1) = (p_2/q_2) = (p_2/q_3) = -1$.

- We define "new" imaginary quadratic number fields k with rank $C_{k,2} = 4$ and infinite 2-class field tower, as fields that to the best of our knowledge were not previously known to have infinite 2-class field tower. With this definition of new fields k, through a generalization of a result by Mouhib (2010) when k is an imaginary quadratic number field with rank $C_{k,2} = 2$, we have demonstrated that there are new fields k in the above family when the 4-rank of C_k is equal to 2, exactly three negative prime discriminants divide d_k, and d_k is congruent to 4 mod 8.

- Through the above generalization of Mouhib's result, we have also shown that there are new fields k when the 4-rank of C_k is equal to 1, d_k is congruent to 4 mod 8, and either five or exactly three negative prime discriminants divide d_k, and infinitely many new fields k when the 4-rank of C_k is equal to 1, exactly three prime discriminants divide d_k, and d_k is not congruent to 4 mod 8.

- Our above results all lend support to the 2-class field tower conjecture. To give an indication of how we have obtained our results, we first state the Golod & Shafarevich Inequality (1964) (as refined by Gaschutz and Vinberg—see Koch (1969)), and two related inequalities used by Sueyoshi (2004, 2009, 2010) that have been derived from a more generic inequality by Martinet (1978).

Lemma 1: Golod & Shafarevich Inequality:

- Let k be a number field, C_k be the class group of k, and E_k be the group of units of k. Then the 2-class field tower of k is infinite if rank $C_{k,2}$ is greater than or equal to $2 + 2(\sqrt{(\text{rank}_2(E_k) + 1)})$, where $\text{rank}_2(E_k)$ is the rank of the elementary 2-group E_k/E_k^2 (and can be described as the number of infinite primes of k).

Lemma 2:

- i) Let F be a totally real number field of degree n, and E be a totally imaginary quadratic extension of F.
  Let t be the number of prime ideals of F which ramify in E. If t ≥ 3 + 2√(n + 1), then the 2-class field tower of E is infinite.

Lemma 2:

- ii) Let F be a totally imaginary number field of degree n, and E be a quadratic extension of F. Let t be the number of prime ideals of F which ramify in E. If $t \geq (n/2) + 3 + 2(\sqrt{(n + 1)})$, then the 2-class field tower of E is infinite.

- We next state the result of Mouhib (2010) in the rank $C_{k,2} = 2$ case that we have generalized to prove our above results.

Lemma 3:

- Let k be an imaginary quadratic number field with $C_{k,2} = 2$ and 4-rank of C_k equal to 1. Let p_1 and p_2 be distinct primes numbers such that the class number (in the wide sense) of $Q(\sqrt{p_1 p_2})$ is divisible by 16. Then for each prime number q congruent to 3 mod 4 such that the Kronecker symbol $(p_1 p_2/q) = -1$, the 2-class field of $Q(\sqrt{-p_1 p_2 q})$ is infinite.

- In order to prove this lemma, Mouhib used the following results from genus theory, that we made use of to generalize Mouhib's above result to our fields k with rank $C_{k,2} = 4$.

Lemma 4:

- Let K be a quadratic extension of a number field k. Then rank $C_{k,2}$ is greater than or equal to ram(K/k) – (dimension over the field F_2 of $E_k/(E_k \cap N_{K/k}(K^*))$ – 1, where ram(K/k) is the number of primes that ramify in the extension K/k, K* is the multiplicative group of K, and $N_{K/k}$ is the norm map in the extension K/k. Furthermore, the dimension over F_2 of $E_k/(E_k \cap N_{K/k}(K^*))$ is less than or equal to [k:Q] if k is totally real, and is less than or equal to (1/2)[k:Q] otherwise, where Q is the field of rational numbers.

Remark 1:

- Recall the well-known result from genus theory that if k is a quadratic number field with discriminant d_k, and t is the number of primes that ramify in k (which is the number of primes that divide d_k), then rank $C_{k,2} = t - 2$ if $d_k > 0$ and is not a sum of two squares, and rank $C_{k,2} = t - 1$ otherwise. To obtain our 4-ranks we utilize the standards technique of d_k-splittings of the second kind, as well as an application of Rédei matrices.

- We now state our generalization of Mouhib's above result, where h(M) refers to the 2-class number of M.

Lemma 5:

- Let p_i, q_i, $i = 1, 2, 3, 4$, be distinct prime numbers such that p_i is congruent to 1 mod 4 and q_i is congruent to 3 mod 4, and let q_5 be a prime number such that q_5 is congruent to 3 mod 4, or $q_5 = 2$ if $q_5{}^* = -4$ or $q_5{}^* = -8$, and q_5 is not equal to q_i for $i = 1, 2, 3$ or 4. Without loss of generality let $M = Q(\sqrt{p_1 p_2 q_1})$ (resp. $Q\sqrt{q_1 q_2 q_3}$), $Q\sqrt{q_1 q_2 q_3 q_4}$), $Q(\sqrt{p_1 p_2 q_1 q_2})$, $Q\sqrt{2 q_1 q_2 q_3}$), $Q(\sqrt{2 p_1 q_1 q_2})$, $Q(\sqrt{2 p_1 p_2 q_1})$, $Q(\sqrt{2 p_1 p_2 p_3})$, $Q(\sqrt{p_1 p_2 p_3 p_4})$). Assume that 16 divides $h(M)$, and $(4 p_1 p_2 q_1 / q_5) = -1$ (resp. $(4 q_1 q_2 q_3 / q_5) = -1$, $(q_1 q_2 q_3 q_4 / q_5) = -1$, $(p_1 p_2 q_1 q_2 / q_5) = -1$, $(2 q_1 q_2 q_3 / q_5) = -1$, $(2 p_1 q_1 q_2 / q_5) = -1$, $(2 p_1 p_2 q_1 / q_5) = -1$, $(2 p_1 p_2 p_3 / q_5) = -1$, $(p_1 p_2 p_3 p_4 / q_5) = -1$). Let L be an imaginary quadratic number field with exactly five primes dividing d_L, and moreover let $L = Q(\sqrt{-p_1 p_2 q_1 q_5})$ (resp. $Q\sqrt{-q_1 q_2 q_3 q_5}$), $Q\sqrt{-q_1 q_2 q_3 q_4 q_5}$), $Q(\sqrt{-p_1 p_2 q_1 q_2 q_5})$, $Q\sqrt{-2 q_1 q_2 q_3 q_5}$), $Q(\sqrt{-2 p_1 q_1 q_2 q_5})$, $Q(\sqrt{-2 p_1 p_2 q_1 q_5})$, $Q(\sqrt{-2 p_1 p_2 p_3 q_5})$, $Q(\sqrt{-2 p_1 p_2 p_3 p_4 q_5})$). Then L has infinite 2-class field tower.

Proof:

- The proof of Lemma 5 is analogous to the proof of Mouhib's above rank $C_{k,2} = 2$ result (Lemma 3). We illustrate the proof for the case $M = Q(\sqrt{p_1 p_2 q_1})$, $L = Q(\sqrt{-p_1 p_2 q_1 q_5})$. From the above remark we see that rank $C_{L,2} = 4$, as $d_L = -4p_1 p_2 q_1 q_5$, and we note that $d_M = 4p_1 p_2 q_1$. We let F be the composite field of M^1 and $Q(\sqrt{-q_5})$, and we see that F is a totally complex quadratic extension of the totally real field M^1, and F/L is unramified. The prime number q_5 is inert in the extension M/Q, since $(p_1 p_2 q_1 / q_5) = -1$, and therefore the q_5-adic place of M is principal and consequently splits completely in M^1. Since the number of q_5-adic places that ramify in F/M^1 is equal to $[M^1:M]$, we obtain ram$(F/M^1) = [M^1:M] + [M^1:Q] = 3[M^1:M]$. Since M^1 is totally real, we know that the dimension of the unit index in Lemma 4, where $k = M^1$ and $K = F$, is less than or equal to $[M^1:Q] = 2[M^1:M]$, and consequently Lemma 4 enables us to conclude that rank $C_{F,2} \geq |M^1:M| - 1 \geq 15$.

- On the other hand, since by the Dirichlet Unit Theorem we know that $\text{rank}_2(E_F) = [M^1:Q] = 2[M^1:M]$, and we can verify that $[M^1:M] - 1 \geq 2 + 2\sqrt{(2|M^1:M| + 1)}$, we see that F satisfies the Golod & Shafarevich Inequality, as described in Lemma 1. It follows that F has infinite 2-class field tower, and since F/L is unramified we are able to conclude that L also has infinite 2-class field tower, which establishes our lemma for this case. The rest of our cases can be proved in an analogous manner to the case for $L = Q(\sqrt{-p_1 p_2 q_1 q_5})$, and we omit the details.

- To show there are infinitely many fields with infinite 2-class field tower in certain cases, we can make use of the Chinese Remainder Theorem (CRT) and Dirichlet's Theorem of Primes in an Arithmetic Progression (DPAP), to establish the following lemma:

Lemma 6:

- Assume there exists a field k that satisfies the conditions of Lemma 5, and let M be the corresponding real quadratic number field given in Lemma 5. Then there exist infinitely many such fields that also satisfy these conditions which have M as its corresponding real quadratic number field, and therefore there exist infinitely many such fields that have infinite 2-class field tower.

Remark 2:

- In the case when d_k is congruent to 4 mod 8, the 4-rank of C_j for a corresponding field j obtained by using the Chinese Remainder Theorem (CRT) and Dirichlet's Theorem of Primes in an Arithmetic Progression (DPAP) may not be the same as the 4-rank of C_k. To see an example of this, let $k = Q(\sqrt{-3.5.7.29}) = Q(\sqrt{-3045})$ and $j = Q(\sqrt{-3.263.5.7.29}) = (\sqrt{-800835})$. It can be readily shown that C_k has 4-rank 1 and C_j has 4-rank 0, and that the field j is obtained from the field k by CRT and DPAP as described in Lemma 6.

- However, if d_k is not congruent to 4 mod 8 then for all fields j in the infinite collection of fields described in Lemma 6, the 4-ranks of C_k and C_j will be the same. It follows that in the case when the 4-rank of C_k is equal to 1, exactly three negative prime discriminants divide d_k, and d_k is not congruent to 4 mod 8, once we obtain a new field through Lemma 5 we are able to use Lemma 6 to obtain infinitely many such new fields.

- We state our above results in the form of the following theorem, where once again k is an imaginary quadratic number field such that rank $C_{k,2} = 4$.

Theorem 1:

- There exist new fields k that satisfy the 2-class field tower conjecture when C_k has 4-rank 2, and when C_k has 4-rank 1 with exactly three negative prime discriminants dividing d_k and d_k congruent (resp. not congruent) to 4 mod 8, or when five negative prime discriminants divide d_k.

- Furthermore, there are infinitely many new such fields k in the case when the 4-rank of C_k is equal to 1, exactly three negative prime discriminants divide d_k, and d_k is not congruent to 4 mod 8.

- We now give some examples of our above results, where we utilize Keith Mathews' number theory website (www.numbertheory.org/php/classnopos.html) to obtain the 2-class numbers of our corresponding real quadratic number fields M.

Example 1:

- k = Q($\sqrt{-5.13.7.827}$) = Q($\sqrt{-376285}$),
  M = Q($\sqrt{5.13.7.827}$) = Q($\sqrt{376285}$);
  then we have h(M) = 16, (5.13.7.827/2) =
  -1, (5/2) = (13/2) = (5/7) = (5/827) =
  (13/7) = (13/827) = -1, C_k has 4-rank 2, and it can
  be readily shown from Lemma 5 and our previous
  results that k is a new field with infinite 2-class field
  tower.

Example 2:

- $k = Q(\sqrt{-23.19.67.3}) = Q(\sqrt{-87837})$;
  $M = Q(\sqrt{23.19.67.3}) = Q(\sqrt{87837})$; then we
  have $h(Q(\sqrt{23.19.67})) = 16$, $(-23/3) = 1$,
  $(-19/3) = (-67/3) = -1$, $(23.19.67/3) = -1$,
  and we see that k is a new field with 4-rank
  of C_k equal to 1 and five negative prime
  discriminants dividing d_k.

Example 3:

- $k = Q(\sqrt{-5.13.7.83}) = Q(\sqrt{-37765})$, $M = Q(\sqrt{5.13.7.83}) = Q(\sqrt{37765})$; then we have $h(M) = 16$, $(5.13.7.83/2) = -1$, and we see that k is a new field with 4-rank of C_k equal to 1, exactly three negative prime discriminants dividing d_k, and d_k congruent to 4 mod 8.

Example 4:

- k = Q($\sqrt{-19.11.191.13.41}$) = Q($\sqrt{-21276827}$), M = Q($\sqrt{19.11.13.41}$) = Q($\sqrt{111397}$); then we have h(M) = 16, (19/191) = (11/191) = (41/191) = -1, (13/191) = 1, (19.11.13.41/191) = -1, and we see that k is a new field with 4-rank of C_k equal to 1, exactly three negative prime discriminants dividing d_k, and d_k not congruent to 4 mod 8. From Lemma 6 and Remark 2 we conclude that there are infinitely many new such fields with infinite 2-class field tower.

Remark 3:

- We note that the method we have described above does not work when C_k has 4-rank 0 (i.e. when $C_{k,2}$ is elementary) because in all such cases we obtain that either the corresponding real quadratic number field M in Lemma 5 is of the form Z/2Z x Z/2Z and therefore we do not have $h(M) \geq 16$, or the kronecker symbol inequality required in Lemma 5 is not satisfied.

- Based upon our above results, we conclude with the following open questions, where once again k is an imaginary quadratic number field such that rank $C_{k,2} = 4$.

Question 1:

- Do there exist infinitely many new fields k that have infinite 2-class field tower when C_k has 4-rank 1 and five negative prime discriminants divide d_k? (from Sueyoshi (2009) we know that d_k must be congruent to 4 mod 8 for such new fields)

Question 2:

- Do there exist infinitely many new fields k that have infinite 2-class field tower when C_k has 4-rank 1 (resp. 4-rank 2), exactly three negative prime discriminants divide d_k, and d_k is congruent to 4 mod 8?

Question 3:

- Do there exist new fields k (resp. infinitely many new fields k) that have infinite 2-class field tower when C_k has 4-rank 0, for any of the following cases:

- A) d_k congruent to 4 mod 8 and exactly three negative prime discriminants divide d_k

- B) d_k not congruent to 4 mod 8 and exactly three negative prime discriminants divide d_k

- C) d_k congruent to 4 mod 8 and five negative prime discriminants divide d_k

- D) d_k not congruent to 4 mod 8 and five negative prime discriminants divide d_k?