Bounds on Height Functions

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Notation

- Let K be a number field.
- $\mathcal{O}_K$ is the ring of integers in K.
- M_K^0 is the set of non-archimedean absolute values which are extended from p-adic absolute values.
- M_K^∞ is the set of archimedean absolute values extended from the usual absolute value.
- $M_K = M_K^0 \cup M_K^\infty$ is the set of all absolute values listed above.

The Height Function over a Number Field

Definition

The *logarithmic height* h over the number field K is a function from the field K to $\mathbb{R}_{\geqslant 0}$ defined as follows

$$h(x) = \frac{1}{[K : \mathbb{Q}]} \left(\sum_{v \in M_K} n_v \log \max\{1, |x|_v\} \right) \text{ for any } x \in K,$$

where n_v is the local degree $[K_v : \mathbb{Q}_v]$.

When $K = \mathbb{Q}$, the height for $\frac{x}{y} \in \mathbb{Q}$ (in lowest terms) is described by

$$h\left(\frac{x}{y}\right) = \log \max\{|x|, |y|\}.$$

Properties of the Height Function

The definition of the height function h can be extended to $\overline{\mathbb{Q}}$, i.e. if $x \in K$ and $x \in K'$, then $h_K(x) = h_{K'}(x)$.

We have the following properties for height functions over the algebraic numbers:

1. If α and $\beta \in K$ are conjugates, then $h(\alpha) = h(\beta)$.
2. (Product formula) For any $x \in K^\times$, we have

$$\prod_{v \in M_K} |x|_v^{n_v} = 1, \text{ where } n_v \text{ is the local degree } [K_v : \mathbb{Q}_v].$$

Important Facts

We have a couple of theorems central to height functions.

Theorem (Northcott)

For any $M, N \in \mathbb{R}_{>0}$, there are only finitely many $\alpha \in \overline{\mathbb{Q}}$ such that

$$h(\alpha) \leqslant M \text{ and } \deg \alpha \leqslant N.$$

Theorem

If $\alpha \in \overline{\mathbb{Q}}^{\times}$, then $h(\alpha) = 0$ if and only if α is a root of unity.

Example

$\left\{0, \pm 1, \pm 2, \pm \frac{1}{2}, \pm 3, \pm \frac{1}{3}, \pm \frac{2}{3}\right\}$ are the only rational x where

$$h(x) \leqslant \log 3.$$

Canonical Height

Definition

We define the n^{th} *iterate of the rational function* φ as follows

$$\varphi^n(x) = (\varphi \circ \varphi \circ \ldots \circ \varphi)(x). \ (n \text{ times})$$

Definition

The canonical height for ϕ is defined as follows

$$\hat{h}_\varphi(x) = \lim_{n \to \infty} \frac{h(\varphi^n(x))}{d^n}, \text{ where } d = \deg \varphi.$$

Two properties about canonical height to note

1. For φ with $\deg \varphi \geqslant 2$, we have $\hat{h}_\varphi(\alpha) = 0$ if and only if α is a preperiodic point of φ or $\alpha = 0$.
2. If $\varphi(x) = x^2$, then $\hat{h}_\varphi = h$, where h is the usual height.

Linear Fractional Transformations

We will focus on *linear fractional transformations* $\varphi(x) = \dfrac{ax+b}{cx+d}$ where a, b, c, and $d \in \mathcal{O}_K$ have no "common factors" and $ad - bc \neq 0$.

No "common factors" means for all $v \in M_K^0$ there exists $\alpha \in \{a, b, c, d\}$ such that $|\alpha|_v = 1$.

For algebraic numbers $x_1, x_2, \ldots, x_n$, we have three properties:

1. $h(x_1 x_2 \cdots x_n) \leqslant h(x_1) + h(x_2) + \cdots + h(x_n)$,
2. $h(x_1 + x_2 + \cdots + x_n) \leqslant h(x_1) + h(x_2) + \cdots + h(x_n) + \log n$, and
3. $h(x_1^{-1}) = h(x_1)$ when $x_1 \neq 0$.

Let's attempt to find an upper bound for

$$h(\varphi(x)) - h(x), \text{ for all } x \in K.$$

Naïve Approach to Bounding Heights

Now, we can use the naïve bounds on heights to find a bound for the expression

$$\begin{aligned} h(\varphi(x)) - h(x) &\leqslant h\left(\frac{ax+b}{cx+d}\right) - h(x), \\ &\leqslant h(ax+b) + h(cx+d) - h(x), \\ &\leqslant h(a) + h(b) + h(c) + h(d) + h(x) + \log 4. \end{aligned}$$

Is there a best possible upper bound (not dependent on $x \in K$)?

A Theorem about a Strict Upper Bound

Theorem (S., 2012)

If $\varphi(x) = \dfrac{ax+b}{cx+d}$ *as defined previously and L is the Galois closure of K, then for all* $\beta \in L$*, we have*

$$h(\varphi(\beta)) - h(\beta) \leqslant \frac{1}{[L:\mathbb{Q}]} \sum_{v \in M_L^\infty} n_v \log \max\{|a|_v + |b|_v, |c|_v + |d|_v\}.$$

This inequality is the "best possible," or strict.

Example

Define $\varphi(x) = \dfrac{-3}{5}x + \dfrac{4}{3} = \dfrac{-9x+20}{15}$. For rational β, we have

$$h(\varphi(\beta)) - h(\beta) \leqslant \log \max\{|15|, |-9| + |20|\} = \log 29.$$

Sketch of Proof

1. First, we get
$$\frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_K^\infty} n_v \log\max\{|a|_v+|b|_v, |c|_v+|d|_v\}$$
as an upper bound. (We do not need to pass to the Galois closure for this part.)
2. Then, we need to find $\beta \in L$ that attains or is infinitesimally close to our upper bound.

An Approximation Theorem

Theorem (Artin-Whaples approximation theorem)

Let $S = \{v_i : 1 \leqslant i \leqslant n\} \subset M_K$ be a finite set of absolute values of K. Let $\beta_1, \ldots, \beta_n \in K$. For any $\varepsilon > 0$, there is $\alpha \in K$ such that

$$|\alpha - \beta_i|_{v_i} < \varepsilon, \text{ for all } i.$$

- Optimally, we would attain the upper bound, but it is difficult or impossible in a number field $K \neq \mathbb{Q}$.
- We use the approximation theorem to find a points that approach the bound infinitesimally.
- We need to pass to the Galois closure L in order to proceed.

After using the Product formula, we have

$$h(\varphi(\beta)) = \frac{1}{[L:\mathbb{Q}]}\left(\sum_{v\in M_L} n_v \log\max\{|a\beta+b|_v, |c\beta+d|_v\}\right).$$

Our goals are to find $\beta \in L$ with two properties

1. $\log\max\{|a\beta+b|_v, |c\beta+d|_v\} = \log\max\{1, |\beta|_v\}$, for all $v \in M_L^0$, and
2. $\log\max\{|a\beta+b|_v, |c\beta+d|_v\}$
   $\approx \log\max\{|a|_v + |b|_v, |c|_v + |d|_v\}$, for all $v \in M_L^\infty$.

We use the Artin-Whaples approximation theorem to achieve both properties for a particular $\beta \in L$.

In the archimedean case, we have for any $x \in L$,

$$\sum_{v \in M_L^\infty} n_v \log \max\{1, |x|_v\} = \sum_{\iota \in \mathrm{Gal}(L/\mathbb{Q})} \log \max\{1, |\iota(x)|\}.$$

Assume $\max\{|a|_v + |b|_v, |c|_v + |d|_v\} = |a|_v + |b|_v$.
To maximize the contribution from the archimedean places, β must be close to the element $\frac{|\iota(a)|}{\iota(a)} \cdot \frac{\iota(b)}{|\iota(b)|}$ on the unit circle.

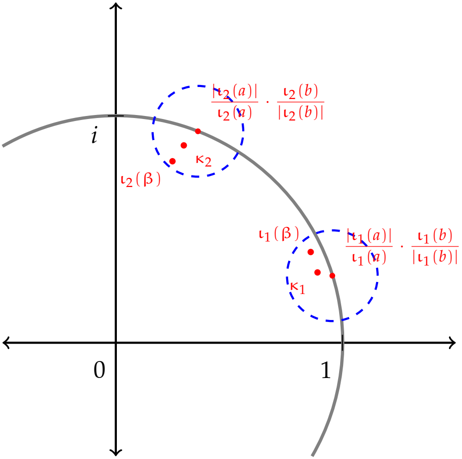
But chances are $\frac{|\iota(a)|}{\iota(a)} \cdot \frac{\iota(b)}{|\iota(b)|}$ is not in L.

Because L is the Galois closure of K, there exists $\kappa_\iota \in L$ such that

$$\left| \frac{|\iota(a)|}{\iota(a)} \cdot \frac{\iota(b)}{|\iota(b)|} - \kappa_\iota \right| < \epsilon,$$

where $\iota(x_\iota) = \kappa_\iota$.

The last condition $|\beta - x_\iota|_v < \epsilon$, for all $v \in M_L^\infty$, can be demonstrated as follows.



An Interesting Corollary

Corollary

In general, if L is the Galois closure of K, then

$$\sup\{h(\varphi(\beta)) - h(\beta) : \beta \in L\} \neq \sup\{h(\beta) - h(\varphi(\beta)) : \beta \in L\}.$$

Proof.

Since φ is a bijection in L, then

$$\begin{aligned}\sup\{h(\varphi^{-1}(\beta)) - h(\varphi(\beta)) : \beta \in L\} &= \sup\{h(\varphi^{-1}(\varphi(\beta))) - h(\varphi(\beta))\} \\ &= \sup\{h(\beta) - h(\varphi(\beta))\}.\end{aligned}$$

We can express the inverse as follows $\varphi^{-1}(x) = \dfrac{dx - b}{-cx + a}$.

So, by our theorem, we have

$$\sup\{h(\beta) - h(\varphi(\beta)) : \beta \in L\}$$
$$= \frac{1}{[L:\mathbb{Q}]} \sum_{v \in M_L^\infty} n_v \log\max\{|a|_v + |c|_v, |b|_v + |d|_v\}.$$

Example

Over $\mathbb{Q}$, Let $\varphi(x) = \dfrac{-9x+20}{15}$ and $\varphi^{-1}(x) = \dfrac{-15x+20}{9}$. By the theorem, we get

$$\sup\{h(\varphi(\beta)) - h(\beta) : \beta \in \mathbb{Q}\} = \log\max\{|15|, |-9| + |20|\} = \log 29;$$
$$\sup\{h(\beta) - h(\varphi(\beta)) : \beta \in \mathbb{Q}\} = \log\max\{|9|, |-15| + |35|\} = \log 35.$$

A theorem by C. Petsche, L. Szpiro, and T. Tucker is as follows.

Theorem

Let σ be a rational map of degree $d \geqslant 2$ in $\mathbb{P}^1(\mathbb{C})$. We have

$$\lim_{n\to\infty} \frac{1}{d^n} \sum_{\sigma^n(\alpha)=\alpha} h(\alpha) = \lim_{n\to\infty} \frac{1}{2^n} \sum_{\zeta^{2^n}=\zeta} \hat{h}_\sigma(\zeta).$$

If we take $\sigma = \varphi^{-1} \circ f \circ \varphi$ where $f(x) = x^2$ is the squaring map and $\varphi(x) = \dfrac{ax+b}{cx+d}$, then after a few steps we can re-write the equation as

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{\sigma^n(\alpha)=\alpha} h(\alpha) = \lim_{n\to\infty} \frac{1}{2^n} \sum_{\zeta^{2^n}=\zeta} h(\varphi(\zeta)).$$