Some results on k-free numbers

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A local point of view (1)

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- Let

$$Q_k(x) := \sum_{n=1}^{x} \mu_k(n),$$

where

$$\mu_k(n) = \begin{cases} 1 & \text{if } n \text{ is } k\text{-free} \\ 0 & \text{if } n \text{ is not } k\text{-free} \end{cases}.$$

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- For $h \in \mathbb{N}$, we define

$$g_k(h) := \max_{x \geq 0} Q_k(x+h) - Q_k(x).$$

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- For some $c_{1,k}, c_{2,k} > 0$ we have

$$c_{1,k}\frac{h^{1/k}}{(\ln h)^{1/k}(\ln\ln h)^{1-1/k}} \le g_k(h) - \frac{h}{\zeta(k)} \le c_{2,k}\frac{h^{2/(k+1)}}{(\ln h)^{2k/(k+1)}}.$$

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- It is unclear what is the exact order for $g_k(h)$. We expect that

$$g_k(h) - \frac{h}{\zeta(k)} \ll h^{1/k+\epsilon}$$

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- Under a plausible conjecture we can show that

$$g_k(h) - \frac{h}{\zeta(k)} \ll h^{3/(2k+1)+\epsilon}$$

for each $\epsilon > 0$.

A local point of view (3)

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- On the opposite direction, we can find gaps of length

$$\frac{\zeta(k)}{k}\frac{\ln x}{\ln\ln x}(1+o(1))$$

as $x \to \infty$. So at least everything in

$$0 \le Q_k(x+h) - Q_k(x) \le \frac{h}{\zeta(k)} + c_{1,k}\frac{h^{1/k}}{(\ln h)^{1/k}(\ln\ln h)^{1-1/k}}$$

happen.

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- We define

$$\mathcal{M}_k(N,h) := \sum_{n=0}^{N-1} \left| \sum_{a=1}^{h} \mu_k(n+a) - \frac{h}{\zeta(k)} \right|^2 .$$

A first global point of view (2)

Theorem. (Hall, L.) *We have*

$$\mathcal{M}_k(N,h) = \gamma_k h^{1/k} N + E_k(h,N)$$

where

$$E_k(h,N) \ll Nh^{1/2k}\exp(-c\sqrt{\ln h}) + Nh^{\theta_k}$$
$$+N^{2/(k+1)}h^{2-2/(k+1)}\ln N + h^2 N^{1/k}\ln N$$

and

$$\gamma_k := 2\frac{\zeta(1/k-1)}{1/k-1}\prod_p\left(1-\frac{1}{p^2}-\frac{2}{p^k}+\frac{2}{p^{k+1}}\right).$$

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- We can take

$$\theta_2 = \frac{11}{53},\ \theta_3 = \frac{1}{6},\ \theta_4 = \frac{1489}{10776},\ \theta_5 = \frac{2513}{21196},\ \theta_6 = \frac{55}{527}$$

$$\theta_k \le \begin{cases} \frac{6}{7k+6} & \text{for } k \in \{7, 8, 9, 10, 11\} \\ \frac{1}{k+3} & \text{for } k \ge 12 \end{cases}.$$

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- With the exponent pairs conjecture, we have

$$\theta_k = \frac{1}{2k+1} + \epsilon$$

for each $\epsilon > 0$.

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- For all but $\ll Nh^{-2\epsilon}$ value of $x \in [0, N] \cap \mathbb{N}$ we have

$$Q_k(x+h) - Q_k(x) = \frac{h}{\zeta(k)} + O(h^{1/2k+\epsilon})$$

for $h \leq N^{\frac{k-1}{2k-1}} / ln^{\frac{k}{2k-1}} N$.

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- We also have the upper bound

$$\mathcal{M}_k(h, N) \ll Nh^{2/k}$$

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- There is a lot of overlapping of intervals in the definition of $\mathcal{M}_k(N, h)$.

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- For $N = h(M+1)$ and for each $\beta \in [0, h-1] \cap \mathbb{N}$ we define

$$\mathcal{T}_k(\beta, h, M) := \sum_{n=0}^{M} \left| \sum_{a=1}^{h} \mu_k(nh + a + \beta) - \frac{h}{\zeta(k)} \right|^2.$$

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- We have the relation

$$\sum_{\beta=0}^{h-1} \mathcal{T}_k(\beta, h, M) = \mathcal{M}_k(N, h).$$

A second global point of view (2)

Theorem. (L.) *For* $N = h(M+1)$ *we have*

$$\mathcal{T}_k(\beta, h, M) = \gamma_k h^{1/k}(M+1) + \mathcal{R}_k(h, N)$$

where

$$\mathcal{R}_k(h, N) \ll Mh^{\theta_k} + Mh^{1/2k} \prod_{p^j || (h, \gamma^k(h))} \left(1 + \frac{2}{p^{k-j}} + \frac{1}{p^{1-j/2k}}\right)$$

$$+h^2 M^{1/k} \ln N + hM^{2/(k+1)} \ln N \prod_{p|h} \left(1 + \frac{2}{p^{k/(k+1)}}\right).$$

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for $h \geq N^{1/3}$.
- We have found a way to cut the interval $[1, N]$ in small subsets and show that the average of μ_k on those subsets is what we expect for most of them. Can we do the same for more general partition of $[1, N]$ like
$$\sum_{n\geq 1}\left|\sum_{a\in I_n}\mu_k(n) - \frac{|I_n|}{\zeta(k)}\right|^2$$
where I_n are intervals with
$$h \leq |I_n| \leq 2h, \quad \bigcup_n I_n = [1, N], \quad I_n \cap I_m = \emptyset \ \forall n \neq m.$$

Arithmetic progression (1)

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- We define

$$\mathcal{W}_k(q,x) := \sum_{a=1}^{q} |E_k(a,q,x)|^2$$

where

$$Q_k(a,q,x) := \sum_{\substack{n=1 \\ n\equiv a \mod q}}^{x} \mu_k(n) =: g_k(a,q)x + E_k(a,q,x)$$

and

$$g_k(a,q) := \frac{1}{q\zeta(k)} \prod_{p|q} \left(1 - \frac{1}{p^k}\right)^{-1} \prod_{\substack{p|q \\ (q,p^k)|a}} \left(1 - \frac{(q,p^k)}{p^k}\right).$$

Arithmetic progression (2)

Theorem. (L.) *For* $x = qM$, *with* $M \geq 1$ *and* $M \in \mathbb{R}$, *we have*

$$\mathcal{W}_K(q, x) = \gamma_k(q) q M^{1/k} + \mathcal{R}_k(q, x)$$

where

$$\begin{aligned}\mathcal{R}_k(q, x) \ll\ & 2^{\omega(q)} q \prod_{p||q} \left(1 + \frac{1}{2p}\right) + I(k\theta_k) M^{1/2k} q \prod_{p|q} \left(1 + \frac{1}{p^{1/2}}\right) \\ & + M^{\theta_k} q \prod_{p|q} \left(1 + \frac{1}{p^{k\theta_k}}\right) + \frac{2^{\omega(q)} x^{1+1/k} \ln x}{q}.\end{aligned}$$

Arithmetic progression (3)

$$
\begin{aligned}
\gamma_k(q) := 2\frac{\zeta(1/k-1)}{1/k-1}\prod_{p\nmid q}\left(1-\frac{1}{p^2}-\frac{2}{p^k}+\frac{2}{p^{k+1}}\right) \\
\times \prod_{\substack{p^\alpha \| q \\ (r-1)k<\alpha\leq rk}} \Bigg(\left(1-\frac{1}{p}\right)\Bigg(\left(1-\frac{1}{p^2}-\frac{2}{p^k}+\frac{2}{p^{k+1}}\right) \\
+\sum_{j=1}^{r-1}\frac{1}{p^{jk}}\left(|\mu(p^j)|-\frac{1}{p}-\frac{1}{p^2}+\frac{2}{p^{k+1}}\right)\Bigg) \\
+\frac{1}{p^{\alpha+r-\alpha/r}}\left(\left[\frac{1}{r}\right]-\frac{1}{p}-\frac{1}{p^2}+\frac{2}{p^{k+1}}\right)\Bigg)
\end{aligned}
$$

Arithmetic progression (4)

Corollary. *For* $1 \leq q \leq x$ *we have*

$$\mathcal{W}_k(q, x) \ll x^{2/k} q^{1-2/k}.$$

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- Counting the solutions to the congruence

$$f(x_1, \dots, x_n) \equiv 0 \mod q$$

for a polynomial $f \in \mathbb{Z}[x_1, \dots, x_n]$ when $(x_1, \dots, x_n) \in [1, x]^n$ with $q \ll x$.

References

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