A q-analog of Fleck's congruence

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Fleck's congruence

In the early 1900's, Fleck proved that for p a prime number and $0 \leq j < p$, one has

$$\sum_{m \equiv j \pmod{p}} (-1)^m \binom{n}{m} \equiv 0 \pmod{p^{\left\lfloor \frac{n-1}{p-1} \right\rfloor}}.$$

Later this was generalized by Sun: for $\alpha \in \mathbb{N}$ and $0 \leq j < p^\alpha$,

$$\sum_{m \equiv j \pmod{p^\alpha}} (-1)^m \binom{n}{m} \equiv 0 \pmod{p^{\left\lfloor \frac{n-p^{\alpha-1}}{\phi(p^\alpha)} \right\rfloor}}.$$

Where Fleck's congruence appears

- Weisman considered $\displaystyle\sum_{m\equiv j \pmod{p^\alpha}} (-1)^m \binom{n}{m}$ to study continuity properties in p-adic analysis

- Wan saw a generalization of Fleck's congruence in working on p-adic L-functions

- Davis and Sun saw it in investigating homotopy p-exponents for $SU(n)$

Our goal

Find a q-analog of Fleck's congruence by investigating divisibility of $\displaystyle\sum_{m\equiv j \pmod p}^{n} (-1)^m \binom{n}{m}_q$

Recall that

$$\binom{n}{m}_q = \begin{cases} \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-m+1})}{(1-q^m)(1-q^{m-1})\cdots(1-q)}, & \text{if } 0 \le m \le n \\ 0, & \text{else} \end{cases}$$

Some basics

Example. $\binom{5}{2}_q = 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6$

Observation. $\binom{n}{m}_{q \to 1} = \binom{n}{m}$

Fact. $\binom{n}{m}_q = \#\left\{m\text{-dimensional subspaces in } \mathbb{F}_q^n\right\}.$

Some basics about q-binomial coefficients

The q-binomial coefficients satisfy some familiar identities:

- $$\binom{n+1}{m}_q = q^m\binom{n}{m}_q + \binom{n}{m-1}_q$$
- $$(1-q^m)\binom{n}{m}_q = (1-q^n)\binom{n-1}{m-1}_q$$

Some more basics about q-binomial coefficients

They also satisfy some less familiar properties:

- $\Phi_r(q) \mid (1 - q^s)$ if and only if $r \mid s$
- $\Phi_n(q) \mid \binom{n}{m}_q$ when $m \neq 0, n$

The Gaussian Formula

The generalization of $\sum_{m=0}^{n} (-1)^m \binom{n}{m} = 0$ is

Theorem (Gaussian Formula)

$$\sum_{m=0}^{n} (-1)^m \binom{n}{m}_q = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \prod_{k \, odd} (1 - q^k) & \text{if } n \text{ is even} \end{cases}$$

Returning to a q-analog of Fleck's congruence

To get started, we did lots of computations

These made it clear that

$$\sum_{m \equiv j \pmod{p}} (-1)^m \binom{n}{m}_q \equiv 0 \pmod{\prod_{k \text{ odd}} \Phi_{kp}(q)^{\epsilon(k)}}$$

where $\epsilon(k) = \left\lfloor \frac{n}{2kp} \right\rfloor$.

Surprisingly, this works more generally than one might guess based on the binomial identity alone.

Theorem (S., Walker '11)

Let $c \in \mathbb{N}$ be given. Then for $0 \leq j < c$, $P \in \mathbb{Z}[x]$ and $z \in \mathbb{N}$ we have

$$\sum_{m \equiv j \pmod{c}} (-1)^{\frac{m-j}{c}} P(m)(q^z)^m \binom{n}{m}_q \equiv 0 \left(\bmod \prod_{k \text{ odd}} \Phi_{kc}^{\epsilon(k,P)}\right)$$

where $\epsilon(k,P) = \left\lfloor \frac{n}{2kc} - \frac{\deg(P)}{2} \right\rfloor$.

Ideas involved in the proof

We begin with a fairly standard trick: expressing a sum over elements in a fixed congruence class by "twisting" sums over all elements.

Proposition

The desired congruence is equivalent to

$$\sum_{m=0}^{n} \zeta_{2c}^{m} P(m)(q^{z})^{m} \binom{n}{m}_{q} \equiv 0 \pmod{\prod_{k \text{ odd}} \Phi_{kc}^{\epsilon(k,P)}}.$$

Relations for various n

$$\sum \zeta_{2c}^m P(m) \binom{n}{m}_q = \sum \zeta_{2c}^m P(m) \left(q^m \binom{n-1}{m}_q + \binom{n-1}{m-1}_q \right)$$

$$= \sum \zeta_{2c}^m P(m) \binom{n-1}{m}_q - \sum \zeta_{2c}^m P(m)(1-q^m) \binom{n-1}{m}_q + \sum \zeta_{2c}^m P(m) \binom{n-1}{m-1}$$

$$= \sum \zeta_{2c}^m P(m) \binom{n-1}{m}_q - (1-q^n) \sum \zeta_{2c}^m P(m) \binom{n-2}{m-1}_q + \sum \zeta_{2c}^m P(m) \binom{n-1}{m-1}$$

Accounting for multiplicities

One can account for additional cyclotomic factors and deal with $\deg(P) > 0$ by using a q-analog of Chu-Vandermonde:

Theorem

$$\sum_{m=0}^{n} \zeta_{2c}^{m} P(m) \binom{n}{m}_q =$$

$$\sum_{j=0}^{kc} \zeta_{2c}^{kc-j} \binom{kc}{j}_q \left(\sum_{m=0}^{n-kc} \zeta_{2c}^{m} P(m+kc-j)(q^j)^m \binom{n-kc}{m}_q \right)$$

...and finally...

Lots and lots of induction.

Alternating sums for even moduli

Fleck's original congruence uses $(-1)^m$ to alternate, so when $p = 2$ this is not an alternating sum.

The q-analog of Fleck's congruence *requires* a bona fide alternating sum.

Forcing alternation in Fleck's congruence when $p = 2$ drops the 2-divisibility by half.

Recovering half of Fleck's congruence

$\Phi_{kc}(1) \neq 1$ if and only if $kc = p^{\alpha}$ for odd prime p or $k = 1$ and $c = 2$.

The number of factors of p when evaluating our q-analog at $c = p$ and $q = 1$ is

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \cdots \approx \frac{1}{2} \left\lfloor \frac{n - p^{\alpha-1}}{\phi(p^{\alpha})} \right\rfloor.$$

Future directions

It seems that this deficiency isn't fault of our theorem; it appears that (at least some) of the cyclotomic factors are sharp

Conjecture

$$\sum_{m\equiv j \pmod c} (-1)^{\frac{m-j}{c}} \binom{(2r+1)c-1}{m}_q \equiv$$

$$(-1)^j \left(\prod_{l=0}^{r-1}(2l+1)\right) \left(-c(q^c-1)\right)^c q^{c-T(j)} \pmod{\Phi_c(q)^{r+1}}.$$

Whence the other factors?

Non-cyclotomic factors of $\displaystyle\sum_{m\equiv 1 \pmod 3} (-1)^{\frac{m-1}{3}} \binom{8}{m}_q$

$$(q^3+q+1)(q^7+q^4+q^3+q-1)$$

Can we interpret these remained polynomials? Can we prove they are highly p-divisible without Fleck's congruence?